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Fermionic Quantum Gravity

Lori D. Paniak

Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544, U.S.A.
paniak@feynman.princeton.edu

Richard J. Szabo

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
szabo@nbi.dk

ABSTRACT

We study the statistical mechanics of random surfaces generated by $N \times N$ one-matrix integrals over anti-commuting variables. These Grassmann-valued matrix models are shown to be equivalent to $N \times N$ unitary versions of generalized Penner matrix models. We explicitly solve for the combinatorics of 't Hooft diagrams of the matrix integral and develop an orthogonal polynomial formulation of the statistical theory. An examination of the large N and double scaling limits of the theory shows that the genus expansion is a Borel summable alternating series which otherwise coincides with two-dimensional quantum gravity in the continuum limit. We demonstrate that the partition functions of these matrix models belong to the relativistic Toda chain integrable hierarchy. The corresponding string equations and Virasoro constraints are derived and used to analyse the generalized KdV flow structure of the continuum limit.

Contents

1 Introduction and Summary

Matrix models [1] provide a useful framework in which to extract quantitative information about complex statistical systems given only their general symmetry properties. The key feature which enables this description is the existence of universal behaviour in these models. Over the years random matrix models have been utilized in a wide range of studies in condensed matter systems [2], quantum chromodynamics [3], and low-dimensional string theory [4, 5]. They are also useful for solving various statistical mechanical models and combinatorial problems [6]. One interpretation of random matrix ensembles is in terms of generators of 't Hooft diagram expansions which correspond to discretizations of compact Riemann surfaces. The continuum limits of such discretizations are realized as phase transitions in the statistical mechanics of the matrix model and characterized by universal data at the critical points.

Despite the success of matrix model technology over the years, there remain several issues in random surface theory which are not adequately addressed by standard approaches. The simplest, and one of the most fundamental issues regards a self-consistent definition of the vacuum of pure two-dimensional quantum gravity. The conventional matrix model methods for investigating this theory lead to inconsistencies. For instance, bosonic matrix integrals are typically ill-defined in the physically interesting region of parameter space corresponding to the non-perturbative continuum limit. Furthermore, they are unable to describe the coupling of gravity to conformal matter fields of central charge $c > 1$. It has been suggested [7] that above this $c = 1$ barrier the two-dimensional geometry degenerates into a tree-like or branched polymer phase.

In this paper we will study a model of discretized random surfaces which is in part motivated by such issues. It is parameterized by a zero-dimensional action involving matrices with Grassmann-valued elements [8]. This model is known as the adjoint fermion one-matrix model and it is defined by the partition function

$$Z_N = \int\limits_{\mathrm{Gr}(N)^c} d\psi \; d\bar{\psi} \; \mathrm{e}^{N \,\mathrm{tr}\, V(\bar{\psi}\psi)} \tag{1.1}$$

where

$$V(\bar{\psi}\psi) = \sum_{k=1}^{K} \frac{g_k}{k} \left(\bar{\psi}\psi\right)^k \tag{1.2}$$

is a polynomial potential of order $K \leq \infty$, and $d\psi \; d\bar{\psi} = \prod_{i,j} d\psi_{ij} \; d\bar{\psi}_{ij}$ is the standard Berezin measure on the complexified Grassmann algebra $\mathrm{Gr}(N)^c = \mathrm{Gr}(N) \otimes_{\mathbb{R}} \mathbb{C}$. For any matrix X the trace is normalized as $\mathrm{tr}\, X = \sum_i X_{ii}$. The matrix elements ψ_{ij} and $\bar{\psi}_{ij}$, $i, j = 1, \ldots, N$, are independent, complex-valued Grassmann numbers with the usual rules for complex conjugation, $\bar{\psi}^*_{ij} = -\psi_{ij}$, $\psi^*_{ij} = -\bar{\psi}_{ij}$, and $(\bar{\psi}_{ij}\psi_{kl})^* = \psi^*_{kl}\bar{\psi}^*_{ij}$ (This ensures that the generating function (1.1) is real-valued). The large N limit of this model has been studied in [8]–[9] and shown to exhibit the range of critical behaviour

seen in the usual bosonic one-matrix models. The topological expansion of the matrix integral (1.1) was considered in [10] and some novel critical behaviour was observed in [11]. The fermionic matrix model can also be coupled to an ordinary complex bosonic one to produce a supersymmetric matrix model, which has been used to describe solutions of certain combinatorial problems [12] and also to generate statistical models of branched polymers [13].

The most attractive feature of the fermionic matrix model (1.1) is that the integration over Grassmann variables is always well-defined and the theory yields finite, computable observables. The dimension N of the Grassmann matrices acts as a cutoff on the number of terms in the partition function (1.1) which counts all fat-graphs generated. The fact that the partition function is a polynomial in the coupling constants g_k for finite N, coupled with the fact that inner products in the Grassmann integration measure are effectively calculable, leads to explicit expressions for statistical quantities in this class of models. In the following we will use this property to solve analytically the combinatorial problem of counting the dynamical triangulations generated by the fermionic partition function via an explicit determinant representation of (1.1).

Since the generating function (1.1) is always a well-defined convergent object at finite N, one might expect that its large N limit is also well-defined. In perturbation theory, the Feynman rules for fermionic matrices include a factor of -1 for each fermion loop thereby resulting in cancellations between large numbers of Feynman diagrams. This observation has led to the conjecture that the genus expansion of the matrix integral (1.1) in the large N limit is an alternating series which may be Borel summable. The evidence for this striking property of the adjoint fermion one-matrix model is its intimate relationship with generalized Penner matrix models [8, 10, 9] which are defined by the Hermitian matrix integrals

$$Z^{\rm H}_{N,\alpha} = \int\limits_{u(N)} d\phi \ {\rm e}^{N\,{\rm tr}\left(V(\phi)+\alpha\log\phi\right)} \tag{1.3}$$

where $d\phi = \prod_{i,j} d\phi_{ij}$ is the Haar measure on the Lie algebra $u(N)$ of the $N \times N$ unitary group $U(N)$. It has been argued that the matrix model (1.3) is equivalent to (1.1) for $\alpha = -2$, with the Hermitian matrix ϕ identified with $\bar\psi\psi$. This has been established in the large N limit by a saddle-point computation on $\phi \sim \bar\psi\psi$ [9], and by showing that their Schwinger-Dyson (loop) equations are identical order by order in the large N expansions of the matrix integrals [8, 10]. However, for $\alpha < 0$ the integration over Hermitian matrices in (1.3) is not well-defined because of the logarithmic divergence at $\phi = 0$. In this region the integral can only be defined by analytical continuation and the matrix model only makes sense at $N = \infty$. This continuation produces complex-valued endpoints for the support of the corresponding distribution of ϕ eigenvalues, analogous to what naturally occurs in fermionic matrix models [8, 9]. It is this complexification of the support of the distribution of eigenvalues that is asserted to result in an alternating genus expansion. Moreover, it can be argued [10] that the topological expansion coincides, modulo these sign

factors, with the usual Painlevé expansions of Hermitian matrix integrals with polynomial potentials [4].

The near equivalence of the genus expansions strongly suggests a correspondence between the Feynman graph expansions in the bosonic and fermionic models. One way of understanding these issues is to appeal to a simplified version of the matrix integral (1.1), namely a fermionic vector model defined by anticommuting vector elements ψ_i and $\bar{\psi}_i$, $i = 1, \ldots, N$ [14]. A fermionic vector model is related to a bosonic vector model, defined with the same polynomial potential V, by a simple mapping of its Feynman diagrams and an analytical continuation $N \mapsto -\frac{N}{2}$ of the vector dimension which results in a bosonic perturbation series with a cutoff on the number of terms generated. This analytical continuation can be understood in terms of a complex-valued saddle-point of the bosonic model in the large N limit, leading to an alternating "genus expansion" which produces a well-defined, unique function [14]. As bosonic vector models generate random models of branched polymers, the fermionic vector model thereby produces a statistical theory of branched polymers whose continuum characteristics are equivalent order by order in large N to the conventional models, but whose double-scaling expansion is a non-perturbatively well-defined function.

The arguments given in [10] are based on the class of models with odd polynomial potential, i.e. $V(-\phi) = -V(\phi)$. In these cases, the endpoints of the support contour of the spectral density are located symmetrically about the origin on the imaginary axis in the complex plane [8, 9]. Generic polynomial potentials are difficult to treat because the corresponding support contours are located asymmetrically in the complex plane and the loop equations become cumbersome to deal with. Moreover, although the equations of motion for the matrix models (1.1) and (1.3) are identical at each order of the large N expansion, beyond the leading order they must be solved with different boundary conditions and the non-perturbative solutions are not the same in the two cases [10]. One of the main results of this paper will be the clarification of the role played by generalized Penner models in fermionic matrix integrals. The analytical expressions for the partition function (1.1) at finite N will be interpreted in terms of Toeplitz determinants, which will naturally lead to a proof of the equivalence of the fermionic one-matrix model and a *unitary* version of the Penner one-matrix model,

$$Z_N^{\mathrm{U}} = \int\limits_{U(N)} [dU]\ \mathrm{e}^{N\,\mathrm{tr}\left(V(U) - \log U\right)} \tag{1.4}$$

where $[dU]$ is the invariant Haar measure on the unitary group $U(N)$. The important feature of this equivalence is that it holds for any matrix dimension N. The finiteness of the fermionic matrix model at finite N is captured by the compactness of the field variable in (1.4). As we will see, the logarithmic term in the effective potential of (1.4) gives certain restrictions on the configurations which contribute to the partition function and its observables when the unitary matrix model is treated from a group theoretic

perspective. These restrictions naturally reproduce the basic characteristics of the matrix integral (1.1), and moreover show precisely how the analytic, functional forms of the models (1.1) and (1.4) are equivalent for any value of N. We will also prove directly that the loop equations of the fermionic and unitary matrix models are identical, and that they admit the same, perturbative Gaussian boundary conditions. It is a remarkable fact that a unitary matrix model such as (1.4) admits an interpretation in terms of random surfaces.

The biggest advantage of the mapping of the fermionic matrix model to a unitary one is that we now have an eigenvalue model to analyse. Grassmann-valued matrices are not diagonalizable, and a direct analysis of fermionic matrix models is only possible using the method of loop equations [8, 9]. In this paper we shall exploit this eigenvalue representation to give a systematic and detailed account of the exotic properties possessed by fermionic matrix models. In particular we will develop a generalization of the orthogonal polynomial technique on the circle which is necessary to study the unitary matrix model defined by (1.4), and describe some of their functional analytic properties. These methods will naturally lead, irrespective of the parity of the potential V, to a precise investigation of the double-scaling limit of the fermionic matrix model which determines a non-perturbative, all-genus continuum theory of two-dimensional quantum gravity coupled to matter. We will show that the fermionic matrix model leads to a Borel summable genus expansion, in contradistinction to Hermitian one-matrix models. Unlike the vector models, the diagrammatic, finite N situation is not quite so simple in the case of fermionic matrices, but the final qualitative result will be the same. In particular, we shall obtain a non-perturbative definition of pure two-dimensional quantum gravity which is equivalent to the usual models order by order in the genus expansion, but which is given by a well-defined unique generating function. This non-perturbative model is what we shall call "fermionic quantum gravity" in the following.

Another advantage of the eigenvalue representation is that it will allow us describe the fermionic one-matrix model (1.1) as an integrable system. Generally, matrix models, being examples of exactly solvable systems, are intimately related to certain reductions of well-known integrable hierarchies of differential equations [15]. In the usual bosonic Hermitian one-matrix models, describing two-dimensional quantum gravity, the partition functions are related to the τ-functions of the integrable Toda chain hierarchy. In the following we will see that the fermionic gravity theory is related to the τ-function of a particular deformation of this hierarchy known as the relativistic Toda chain. We can therefore understand many of the novel aspects of the fermionic theory, such as what sort of matter coupling it involves, in terms of this deformation. Furthermore, we will develop an operator theoretic approach to the equations of motion of the matrix integral (1.1) and gain in this setting a precise picture of how this deformation leads to a well-defined Borel summable partition function of the statistical model. This sets the stage for a description of the integrable differential hierarchies satisfied by the continuum theory

which generalizes the usual KdV flow structure of gravity [4, 5]. It will also indicate that the partition function (1.1) in the continuum limit serves as a concrete, non-perturbative definition of two-dimensional quantum gravity.

The organization of the remainder of this paper is as follows. In section 2 we will derive an analytical expression for the partition function (1.1) and use it to describe the solution to the problem of counting fermionic fat-graphs. We shall see that the matrix integral generates a novel class of discretizations of Riemann surfaces whose polygons are two-colourable. We will also briefly describe what sort of matter-coupled gravity theory is represented by the fermionic one-matrix model, and prove its equivalence to the unitary one-matrix model (1.4). In section 3 we will present the formal orthogonal polynomial solution of the fermionic matrix model. In section 4 we will describe the large N limit of the model from the perspective of these orthogonal polynomials. We will find that the adjoint fermion one-matrix model actually possesses a sort of "internal" branched polymer critical behaviour which can be understood in terms of the fermionic characteristics of its fat-graph combinatorics. We shall also demonstrate that the usual universality classes of two-dimensional gravity arise, and that they cannot be smoothly connected to the branched polymer phases of the theory. Of course, the polymer effect is washed out at $N = \infty$ because the surface effect is of higher order in N. We also construct the genus expansion of the theory and establish its Borel summability explicitly. In section 5 we present the operator approach to the fermionic matrix model and use it to interpret the partition function as a τ-function. We derive the Virasoro constraints satisfied by this τ-function, which at the same time establishes explicitly the equivalence of the loop equations of the matrix models (1.1), (1.3) with $\alpha = -2$, and (1.4), in the large N expansion. We then use these descriptions to give an functional analytic explanation for the nature of the topological expansion in the fermionic one-matrix model. Finally, in appendix A we describe properties of observables of the statistical model (1.1) using the unitary matrix model and its orthogonal polynomials, while in appendix B we show that the orthogonal polynomials of the Gaussian fermionic one-matrix model are given in terms of confluent hypergeometric polynomials.

2 Determinant Representations of the Partition Function

In this section we will present an analytic solution to the problem of counting fermionic fat-graphs. We will obtain a closed expression for the generating function of such discretized random surfaces, and at the same time we will be naturally led to the equivalence of the fermionic matrix model and the unitary matrix model which will occupy the bulk of our analysis in subsequent sections. We will also obtain in this way a geometric picture of what sort of theory of random surfaces is represented by the adjoint fermion one-matrix

model and also of the precise role of the corresponding Penner interaction.

2.1 Hermitian Representation

One intriguing feature of the fermionic one-matrix model (1.1) is the extent to which it can be solved at finite-N. This is a consequence of the convergence of the integration over Grassmann variables, in contrast to the case of bosonic matrix models. The combinatorics of the fat-graph expansion of the fermionic partition function Z_N can be determined by using the observation of [11] that the partition function and observables of the matrix model (1.1) can be evaluated explicitly by introducing two $N \times N$ Hermitian matrices ϕ and λ defined by

$$1 = \int\limits_{u(N)} d\phi\ \delta(\phi - \bar\psi\psi) \qquad , \qquad \delta(\phi) = \int\limits_{u(N)} \frac{d\lambda}{(2\pi)^{N^2}}\ \mathrm{e}^{i\,\mathrm{tr}\,\lambda\phi} \tag{2.1}$$

By inserting these definitions into (1.1), the fermionic partition function can be written as

$$\begin{aligned} Z_N &= \int\limits_{u(N)} d\phi\ \mathrm{e}^{N\,\mathrm{tr}\,V(\phi)} \int\limits_{\mathrm{Gr}(N)^c} d\psi\ d\bar\psi \int\limits_{u(N)} \frac{d\lambda}{(2\pi)^{N^2}}\ \mathrm{e}^{i\,\mathrm{tr}\,\lambda(\phi-\bar\psi\psi)} \\ &= \frac{1}{(2\pi)^{N^2}} \iint\limits_{u(N)} d\phi\ d\lambda\ \mathrm{e}^{N\,\mathrm{tr}\,V(\phi)+i\,\mathrm{tr}\,\lambda\phi}\ \det{}^N(-i\lambda) \\ &= \int\limits_{u(N)} d\phi\ \mathrm{e}^{N\,\mathrm{tr}\,V(\phi)}\ \det{}^N\left(-\tfrac{\partial}{\partial\phi}\right)\delta(\phi) \end{aligned} \tag{2.2}$$

so that

$$Z_N = \det{}^N\left(\tfrac{\partial}{\partial\phi}\right)\ \mathrm{e}^{N\,\mathrm{tr}\,V(\phi)}\ \Big|_{\phi=0} \tag{2.3}$$

A similar expression can also be derived for the correlators of the fermionic one-matrix model [11, 9]. However, it is difficult to write down an explicit expression for (2.3) which is informative and amenable to analysis.

To evaluate the determinant in (2.3) explicitly, we shall instead write the fermionic matrix model as a random theory of the two Hermitian matrices ϕ and λ introduced above, keeping only the second line of the identity (2.2). In this way, the fermionic one-matrix model can be written as the Hermitian two-matrix model

$$Z_N = \frac{N^{N(N+1)}}{(2\pi)^{N^2}} \iint\limits_{u(N)} d\phi\ d\lambda\ \mathrm{e}^{N\,\mathrm{tr}\left(V(\phi)+\log(-i\lambda)+i\lambda\phi\right)} \tag{2.4}$$

where we have rescaled $\lambda_{ij} \to N \cdot \lambda_{ij}$. The doubling of degrees of freedom is required to compensate for both the matrix ψ and its adjoint $\bar\psi$. The Hermitian two-matrix model (2.4) describes two-dimensional discretized gravity with a single type of matter state at each of the vertices. It does not admit the usual $\mathbb{Z}_2$ Ising symmetry of a classical spin

lattice that characterizes the unitary minimal models of rational conformal field theory [4]. Thus the matter states in the worldsheet interpretation of the fermionic matrix model are more complicated degrees of freedom than just simple Ising spins or other conventional types of conformal matter. The logarithmic nature of the effective two-matrix potential in (2.4) is just another reflection of the connection with Penner matrix models. The original Penner model [16] ($g_k = 0$ for $k > 1$, $g_1 = -1$ and $\alpha = 1$ in (1.3)) we used to calculate the virtual Euler characteristics of the moduli spaces of compact Riemann surfaces. For $\alpha > 0$ the generalized models are intimately related to the coupling of gravity to matter fields of central charge $c = 1$ [10]. Note that the Penner interaction in (2.4) is well-defined. We may refer to the fermionic matrix model as a model of two-dimensional quantum gravity interacting with "Penner matter". As we will see later on, it is this worldsheet model which leads to a well-defined nonperturbative theory of quantum gravity in two-dimensions which we will call "fermionic gravity".

The partition function (2.4) can be written in terms of a double eigenvalue distribution by diagonalizing the Hermitian matrices $\phi = U\,\mathrm{diag}(\phi_1,\dots,\phi_N)\,U^\dagger$ and $\lambda = V\,\mathrm{diag}(\lambda_1,\dots,\lambda_N)\,V^\dagger$ by unitary transformations, where $\phi_i,\lambda_i \in \mathbb{R}$ are the eigenvalues of ϕ,λ. Computing the Jacobian for the change of integration measure and using the Itzykson-Zuber formula [17] to integrate out the unitary degrees of freedom, we arrive at the eigenvalue model

$$Z_N = c_N \prod_{i=1}^{N} \iint_{-\infty}^{\infty} d\phi_i\; d\lambda_i\; \lambda_i^N\; \mathrm{e}^{NV(\phi_i)+iN\lambda_i\phi_i}\; \Delta(\phi_1,\dots,\phi_N)\,\Delta(\lambda_1,\dots,\lambda_N) \tag{2.5}$$

where $c_N = (-iN^2)^{N(N+1)/2}/(2\pi)^N \prod_{n=1}^{N} n!$ and

$$\Delta(x_1,\dots,x_N) = \det_{i,j}\left[x_i^{j-1}\right] = \prod_{i<j}(x_i - x_j) \tag{2.6}$$

is the Vandermonde determinant. Using (2.6) we can then write the partition function as

$$Z_N = c_N N!\;\det_{i,j}\left[\left(\phi^{i-1}\,,\,\lambda^{j-1}\right)\right] \tag{2.7}$$

where we have introduced the inner product

$$\left(F(\phi)\,,\,G(\lambda)\right) \equiv \iint_{-\infty}^{\infty} d\phi\; d\lambda\; \lambda^N\; \mathrm{e}^{NV(\phi)+iN\lambda\phi}\; F(\phi)\,G(\lambda) \tag{2.8}$$

on the vector space $\mathbb{C}[\phi]\otimes\mathbb{C}[\lambda]$ of complex coefficient polynomials in (ϕ,λ). A remarkable feature of the fermionic matrix model is the extent to which its inner product (2.8) can be computed. For arbitrary polynomial potential, integrating over λ first in (2.8) leads to the result

$$\left(F(\phi)\,,\,\lambda^k\right) = \frac{2\pi}{N}\,\left(\frac{i}{N}\,\frac{\partial}{\partial\phi}\right)^{N+k}\,\left[F(\phi)\;\mathrm{e}^{NV(\phi)}\right]\,\Big|_{\phi=0} \tag{2.9}$$

which is valid for any analytic function $F(\phi)$ and any integer $k \geq 0$. Using (2.9), the representation (2.7) of the partition function as the determinant of the moment matrix of the inner product (2.8) thereby yields

$$Z_N = (-1)^{[N]_2} \det_{i,j} \left[\left(\frac{\partial}{\partial \phi} + NV'(\phi) \right)^{N+j-i} \cdot 1 \, \Big|_{\phi=0} \right] \tag{2.10}$$

where, for any integer $K \geq 1$, $[N]_K$ denotes the largest integer less than or equal to N/K. Here and in the following a prime denotes differentiation. The expression (2.10) is the desired explicit expansion of the determinant operator in (2.3).

2.2 Combinatorics of Fermionic Ribbon Graphs

In contrast to the determinant form (2.3), the representation (2.10) leads to an explicit expression for the perturbation series of the fermionic one-matrix model. For a polynomial potential (1.2) of order K, the determinant in (2.10) may be evaluated explicitly by expanding the exponential function in its Taylor series and applying the multinomial theorem to each term in the expansion to get

$$\begin{aligned} \frac{\partial^L}{\partial \phi^L} \, \mathrm{e}^{NV(\phi)} \, \Big|_{\phi=0} \; &= \; (Ng_1)^L \sum_{k_2=0}^{[L]_2} \cdots \sum_{k_K=0}^{[L]_K} \frac{L!}{(L-2k_2-\cdots-Kk_K)!\, k_2! \cdots k_K!} \\ &\quad \times \prod_{\ell=2}^{K} \left(\frac{g_\ell}{\ell N^{\ell-1} (g_1)^\ell} \right)^{k_\ell} \end{aligned} \tag{2.11}$$

for any integer $L \geq 0$. Normalizing by the Gaussian model for which $V(\phi) = g_1 \phi$ and $Z_N^{\mathrm{Gauss}} = (-1)^{[N]_2} (Ng_1)^{N^2}$, the partition function (2.10) may be written as

$$\frac{Z_N}{Z_N^{\mathrm{Gauss}}} = \det_{i,j} \left[\sum_{k_2=0}^{[N+j-i]_2} \cdots \sum_{k_K=0}^{[N+j-i]_K} \frac{(N+j-i)!}{(N+j-i-\sum_\ell \ell k_\ell)!\, k_2! \cdots k_K!} \prod_{\ell=2}^{K} \left(\frac{g_\ell}{\ell N^{\ell-1} (g_1)^\ell} \right)^{k_\ell} \right] \tag{2.12}$$

We can extract the sums in (2.12) out of the determinant by using the multilinearity of the determinant as a function of its N rows to get

$$\begin{aligned} \frac{Z_N}{Z_N^{\mathrm{Gauss}}} \; &= \; \sum_{k_2^{(1)},\ldots,k_2^{(N)} \geq 0} \cdots \sum_{k_K^{(1)},\ldots,k_K^{(N)} \geq 0} \left[\prod_{m=2}^{K} \prod_{n=1}^{N} \frac{1}{k_m^{(n)}!} \left(\frac{g_m}{m N^{m-1} (g_1)^m} \right)^{k_m^{(n)}} \right] \\ &\quad \times \det_{i,j} \left[\frac{(N+j-i)!}{\left(N+j-i-\sum_\ell \ell k_\ell^{(i)}\right)!} \prod_{\ell=2}^{K} \Theta\left([N+j-i]_\ell - k_\ell^{(i)} \right) \right] \end{aligned} \tag{2.13}$$

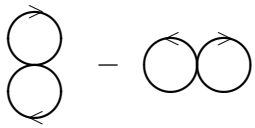
where Θ denotes the step function with the convention $\Theta(0) = 1$.

The expression (2.13) leads to a relatively straightforward expansion of the partition function in powers of the coupling constants g_ℓ, $\ell = 2, \ldots, K$. It represents the formal solution to the problem of counting the (connected and disconnected) Feynman-'t Hooft

diagrams of the adjoint fermion one-matrix model. It is a closed formula, in terms of a sum over a large set of integers, for the generating function of ribbon graphs of the fermionic matrix model. A fermionic ribbon graph has much more structure to it than a conventional fat-graph, and so we shall now formulate the rules for generating them from the matrix integral. The fermionic propagator is $\langle\bar{\psi}_{ij}\psi_{kl}\rangle_{\rm Gauss} = \delta_{il}\,\delta_{jk}/Ng_1$ which we represent in the usual way as a double line, along with an orientation defined by an arrow pointing towards the $\bar{\psi}$ matrix. Vertices are likewise given an orientation by assigning an outgoing arrow for a $\bar{\psi}$ line and an incoming arrow for a ψ line. Fat-graphs are now drawn with the rule that only $\bar{\psi}$ and ψ lines can contract. Each such graph is dual to some discretization of a Riemann surface in the standard way. The surface may be two-coloured by shading the triangles in a polygon if and only if their outer edge is dual to a propagator with an incoming arrow. In this way black edges are always associated with $\bar{\psi}$ matrices and white ones with ψ. Since the fermionic propagator only connects $\bar{\psi}$'s with ψ's, it follows that every such discretization can be two-coloured. Thus the fat-graphs generated by the fermionic matrix integral can be obtained by drawing all discretizations associated with fermionic k-point vertices in terms of the corresponding Hermitian $2k$-point vertices, and keeping only those graphs which are two-colourable. The even parity of the Hermitian potential $V(\phi^2)$ is required because only discretizations with even-sided polygons can be two-coloured.

However, the Grassmann nature of the generating matrices yields important changes to the rules for counting such triangulations. For a given vertex in the Wick expansion of the matrix integral, we fix a chosen $\bar{\psi}$ line. Lines are joined into propagators with a left-handed orientation. Every time a ψ line is joined into a $\bar{\psi}$ line (rather than the other way around), we "twist" the ribbon and thereby reverse its orientation. To each such twisted line, we associate a factor of -1. This standard sign factor for fermionic fields yields significant reductions in the overall number of ribbon graphs which are actually generated by the fermionic matrix model, because of the cancellations which occur, for example, between fermionic diagrams that are topologically the same but which are twisted relative to one another. Thus topologically equivalent diagrams do not necessarily add up, but may have the effect of cancelling each other. As a simple example, consider the quadratic potential, $K = 2$. There are two graphs which contain only a single four-point vertex, but they contribute with equal magnitude and opposite sign to the generating function. This vanishing contribution is the coefficient of the $g_2/2N(g_1)^2$ term in the perturbative expansion of the partition function. Symbolically, it is given by the Wick expansion of the Gaussian correlator

$$\left\langle {\rm tr}(\bar{\psi}\psi)^2\right\rangle_{\rm Gauss} \;=\; \;-\; \;=\; 0 \tag{2.14}$$



Note the twist in the second propagator of the second graph in (2.14), which induces the relative minus sign. Notice also that the usual toroidal four-point graph does not appear

in (2.14), because it cannot be two-coloured. This vanishing contribution is also easily found from the analytical formula (2.13). The leading term is 1 and it comes from the configuration whereby all k's are 0. The terms of order g_2 come from the configuration whereby only one k is non-vanishing and equal to unity. In the determinant in (2.13), the only Θ's which are not equal to unity are those which appear in row i with $k^{(i)} = 1$. The determinant therefore vanishes, reproducing the graphical result (2.14). The remaining fat-graph combinatorics are now readily determined from the graphs of the Hermitian one-matrix model with even potential $V(\phi^2)$ by keeping only those which are two-colourable and incorporating the appropriate twists. The determinant formula (2.13) is readily seen to reproduce the correct combinatorics, with appropriate minus signs coming from the determinant.

These arguments readily generalize to arbitrary polynomial potentials. Generally, the total number of 2ℓ-valent vertices in a given ribbon graph Γ as determined by the formula (2.13) is the power of g_ℓ which is given by

$$n_{2\ell}(\Gamma) = \sum_{n=1}^{N} k_\ell^{(n)} \tag{2.15}$$

and, because of Fermi statistics, this number is bounded as

$$0 \leq n_{2\ell}(\Gamma) \leq \sum_{n=1}^{N} [2N - n]_\ell \tag{2.16}$$

It follows that there are only a finite number of fermionic fat-graphs generated. The twisting of ribbons and also the finiteness of the perturbation series may be attributed to the inclusion of a logarithmic interaction $-2\log\phi$ in the corresponding Hermitian one-matrix model with potential $V(\phi^2)$. This role of the Penner potential will be demonstrated explicitly in the next subsection.

The determinant expansion (2.13) is therefore an analytic expression of the combinatorial formula

$$\frac{Z_N}{Z_N^{\text{Gauss}}} = \sum_{\Gamma} \frac{(-1)^{t(\Gamma)}\,(Ng_1)^{\chi(\Gamma)}}{|\text{Aut}(\Gamma)|} \prod_{\ell=2}^{K} \left(\frac{g_\ell}{g_1}\right)^{n_{2\ell}(\Gamma)} \tag{2.17}$$

where the sum runs through all (connected and disconnected) two-colourable fat-graphs Γ with $n_{2\ell}(\Gamma)$ vertices of order 2ℓ bounded as in (2.16), and $\chi(\Gamma)$ is the Euler characteristic of Γ. The alternating sign factor in (2.17), with $t(\Gamma)$ the total number of twisted ribbons of Γ, is due to Fermi statistics, while $\text{Aut}(\Gamma)$ is the automorphism group of the unmarked graph Γ. The sum over fat-graphs in (2.17) is finite, since the maximum number of vertices that a given diagram can have is

$$v_{\max} = \sum_{\ell=2}^{K} \sum_{n=1}^{N} [2N - n]_\ell \tag{2.18}$$

Each such ribbon graph is dual to a tessellation Γ^* of a Riemann surface of Euler characteristic $\chi(\Gamma)$ by $n_{2\ell}(\Gamma)$ 2ℓ-valent tiles for $\ell = 2, \ldots, K$. Because of the twist factors,

the overall combinatorial numbers generated by the fermionic matrix integral will be drastically reduced. Moreover, depending on the precise details of the potential V, the perturbative or topological expansions of the matrix model may be alternating series. The above considerations show that there is clearly no simple mapping between the Hermitian and fermionic matrix models. These features will have remarkable implications later on for the topological expansion of the matrix integral (1.1).

2.3 Unitary Representation

We will now derive an alternative "dual" representation of the perturbative expansion of the fermionic partition function. For this, we note that the derivatives in (2.9) can be evaluated in terms of a contour integration about the origin of the complex plane as

$$\Big(F(\phi)\,,\,\lambda^k\Big) = -\left(\frac{i}{N}\right)^{N+k+1}\,(N+k)!\oint\limits_{z=0} dz\;\frac{F(z)\;\mathrm{e}^{NV(z)}}{z^{N+k+1}} \tag{2.19}$$

The partition function (2.10) may then be written as

$$Z_N = c_N N!\prod_{k=1}^{N}\left(-\left(\frac{i}{N}\right)^{N+k}\,(N+k-1)!\right)\;\det A \tag{2.20}$$

where we have defined the $N\times N$ matrix

$$A_{ij} = \oint\limits_{z=0} dz\;\frac{\mathrm{e}^{NV(z)}}{z^{N+j-i+1}} \tag{2.21}$$

The crucial observation now is that the matrix elements (2.21) depend only on the differences $j-i$ of row and column positions, i.e. $A_{ij}=A_{j-i}$. Matrices with such a symmetry are known as Toeplitz matrices and their determinants can be evaluated in terms of averages over the unitary group [18]. The Toeplitz determinant $\det_{i,j}[A_{j-i}]$ appearing in (2.20) is associated with the Laurent series of a function $\mathcal{A}$ through the definition

$$\mathcal{A}(\omega) = \sum_{n=-\infty}^{\infty} A_n\,\omega^n \tag{2.22}$$

The matrix elements A_n are then interpreted as the Fourier coefficients of $\mathcal{A}$ on the unit circle $|\omega|=1$,

$$A_n = \oint\frac{d\omega}{2\pi i\omega}\;\omega^{-n}\,\mathcal{A}(\omega) \tag{2.23}$$

Using the fact that a determinant is a linear function of each of its rows, it follows that

$$\det A = \prod_{k=1}^{N}\oint\frac{d\omega_k}{2\pi i\omega_k}\;\mathcal{A}(\omega_k)\;\omega_k^k\;\overline{\Delta(\omega_1,\ldots,\omega_N)} \tag{2.24}$$

Replacing the right-hand side of (2.24) by its average over all permutations of the integration variables ω_k yields

$$\det A = \frac{1}{N!}\prod_{k=1}^{N}\oint\frac{d\omega_k}{2\pi i\omega_k}\;\mathcal{A}(\omega_k)\;\Big|\Delta(\omega_1,\ldots,\omega_N)\Big|^2 \tag{2.25}$$

which shows that the Toeplitz determinant is given by the unitary matrix integral

$$\det A = \frac{\prod_{n=1}^{N-1} n!}{(2\pi)^{N(N-1)/2}} \int\limits_{U(N)} [dU] \ \det \mathcal{A}(U) \tag{2.26}$$

Upon substituting (2.21) into (2.22), we can interchange the sum and contour integration provided we take into account that the latter vanishes for $N + n < 0$. We then have

$$\mathcal{A}(\omega) = \oint\limits_{z=0} dz \ \frac{\mathrm{e}^{NV(z)}}{\omega^N (z-\omega)} = 2\pi i \ \mathrm{e}^{N\left(V(\omega) - \log \omega\right)} \tag{2.27}$$

From (2.20), (2.26) and (2.27) it follows that the partition function of the adjoint fermion one-matrix model (1.1) is completely equivalent, for any dimension N, to the $N \times N$ unitary matrix model

$$Z_N = k_N \int\limits_{U(N)} [dU] \ \mathrm{e}^{N \,\mathrm{tr}\left(V(U) - \log U\right)} \tag{2.28}$$

where $k_N = N^{N(N+1)/2}/(2\pi)^{N(N-1)/2} \prod_{n=1}^{N} (N+n-1)!$. We see that the fermionic one-matrix model for any N is completely equivalent to both a Hermitian two-matrix model, and a unitary one-matrix model, both of which involve Penner-type interactions. In particular, the unitary representation clearly demonstrates that the equivalence with the Hermitian Penner one-matrix model is not true at finite N, but rather only at $N = \infty$. Note that the matrix integral (2.28) is perfectly well-defined at finite N, but it involves an integrand which is not a real-valued potential.

With the partition function given as an integral over the unitary group, we can clarify the geometric role of the Penner-type potential that characterizes fermionic matrix models, and also give an alternative method for a perturbative expansion in terms of the coupling constants in the potential V. For this, we integrate over the $U(1)$ factor of the unitary group $U(N) = U(1) \times SU(N)/\mathbb{Z}_N$ and leave an $SU(N)$ integral,

$$Z_N = k_N \int\limits_{U(N)} d\theta \, [dU_0] \ \mathrm{e}^{N\left(\mathrm{tr}\, V(\mathrm{e}^{i\theta} U_0) - iN\theta\right)} = \frac{k_N}{N} \int\limits_0^{2\pi} d\theta \ \mathrm{e}^{-iN^2\theta} \int\limits_{SU(N)} [dU_0] \ \mathrm{e}^{N \,\mathrm{tr}\, V(\mathrm{e}^{i\theta} U_0)} \tag{2.29}$$

The role of the $U(1)$ integration over θ is to restrict the terms in the $SU(N)$ integration to those involving only N^2 factors of U_0. Equivalently, we may expand the invariant function which is the integrand of (2.29) in $U(N)$ characters $\chi_{\vec{n}}$ as

$$\mathrm{e}^{N \,\mathrm{tr}\, V(U)} = \sum_{\vec{n}} c_{\vec{n}} \ \chi_{\vec{n}}(U) \qquad , \qquad U = \mathrm{e}^{i\theta} U_0 \tag{2.30}$$

where $c_{\vec{n}} \in \mathbb{C}$ are functions of the coupling constants of the potential V, and the sum goes over all irreducible polynomial representations of $U(N)$ which may be parameterized by their highest weight components $n_1 \geq \ldots \geq n_N \geq 0$ associated with the lengths of the lines in the corresponding Young tableaux. Since

$$\chi_{\vec{n}}(U) = \mathrm{e}^{i\theta C_1(\vec{n})} \, \chi_{\vec{n}}(U_0) \tag{2.31}$$

where

$$C_1(\vec{n}) = \sum_{k=1}^{N} n_k \tag{2.32}$$

is the linear Casimir of the $U(N)$ representation $\vec{n}$ (the total number of boxes in the Young tableau), it follows that the constraint on the $SU(N)$ integration in (2.29) is a restriction to terms with linear $U(N)$ Casimir C_1 equal to N^2. We may therefore write

$$Z_N = \frac{k_N}{N} \left[\int\limits_{SU(N)} [dU_0]\ \mathrm{e}^{N\,\mathrm{tr}\,V(U_0)} \right]_{C_1=N^2} \tag{2.33}$$

where the bracket means to restrict the character expansion of $\mathrm{e}^{N\,\mathrm{tr}\,V(U_0)}$ to those $\vec{n}$ with $C_1(\vec{n}) = N^2$. The role of the logarithmic interaction in (2.28) may therefore be characterized as restricting the matrix integral to completely filled Young diagrams.

The $SU(N)$ integral in (2.33) can be evaluated using standard techniques from lattice gauge theory [19]. For this, we introduce the generating function

$$W(J) = \int\limits_{SU(N)} [dU_0]\ \mathrm{e}^{\mathrm{tr}\,J\,U_0} \tag{2.34}$$

with the property

$$\int\limits_{SU(N)} [dU_0]\ f(U_0) = f\left(\tfrac{\partial}{\partial J}\right) W(J) \Big|_{J=0} \tag{2.35}$$

where J is an arbitrary $N \times N$ matrix and f an arbitrary function on $SU(N)$. Using the invariance of the Haar measure $[dU_0]$ under $SU(N)$ rotations, we have $W(J) = W(UJV)$, $U, V \in SU(N)$, from which it can be shown that the generating function may be expanded in powers of determinants of J as [19]

$$W(J) = \frac{(2\pi)^{\frac{N(N+1)}{2}-1}}{N} \sum_{k=0}^{\infty} \frac{(\det J)^k}{\prod_{n=1}^{N}(n+k-1)!} \tag{2.36}$$

Applying this result to the partition function (2.33), the restriction $C_1(\vec{n}) = N^2$ implies that only the $k = N$ term of the expansion (2.36) contributes, giving

$$Z_N = \mathrm{e}^{N\,\mathrm{tr}\,V\left(\frac{\partial}{\partial J}\right)}\ \det{}^N J \Big|_{J=0} \tag{2.37}$$

Comparing (2.3) and (2.37), we see that the finite N expressions for the fermionic partition function in the Hermitian two-matrix and unitary one-matrix representations are dual to each other. We can make this duality precise by writing (2.37) as

$$\begin{aligned} Z_N &= \frac{1}{(2\pi)^{N^2}} \iint\limits_{gl(N)} dJ\ d\phi\ \mathrm{e}^{i\,\mathrm{tr}\,\phi J}\ \mathrm{e}^{N\,\mathrm{tr}\,V\left(\frac{\partial}{\partial J}\right)}\ \det{}^N J \\ &= -\frac{1}{(2\pi)^{N^2}} \iint\limits_{gl(N)} dJ\ d\phi\ \mathrm{e}^{N\,\mathrm{tr}\,V(i\phi)}\ \det{}^N\left(-i\,\tfrac{\partial}{\partial \phi}\right)\ \mathrm{e}^{i\,\mathrm{tr}\,\phi J} \\ &= \int\limits_{gl(N)} d\phi\ \delta(\phi)\ \det{}^N\left(-i\,\tfrac{\partial}{\partial \phi}\right)\ \mathrm{e}^{N\,\mathrm{tr}\,V(i\phi)} \;=\; \det{}^N\left(\tfrac{\partial}{\partial \phi}\right)\ \mathrm{e}^{N\,\mathrm{tr}\,V(\phi)} \Big|_{\phi=0} \end{aligned} \tag{2.38}$$

Some further aspects of the unitary representation of the fermionic matrix model are discussed in appendix A. In section 5 we will show explicitly that the equations of motion of the unitary and fermionic one-matrix models are identical.

3 Orthogonal Polynomial Solution

In this section we shall demonstrate how to generalize the orthogonal polynomial technique [20] to the adjoint fermion one-matrix model (1.1), with the goal of solving the matrix model in the large N limit. Although the fermionic matrix model is not naturally an eigenvalue model, since it is not possible to diagonalize a Grassmann valued matrix, we can exploit its equivalence with the unitary one-matrix model (2.28). We will thereby *define* the orthogonal polynomials of the fermionic matrix model (1.1) to be the orthogonal polynomials associated with the corresponding unitary eigenvalue model. This definition will lead, as we shall see, to a well-defined real-valued solution for the fermionic matrix model.

3.1 General Properties

Let us consider the fermionic partition function in the representation (2.20,2.21), which involves the measure

$$d\mu(z) = \frac{\mathrm{e}^{NV(z)}}{z^{N+1}}\ dz \tag{3.1}$$

on the unit circle $z\bar{z} = 1$. The partition function (2.20) may then be written as the determinant of the moment matrix of the measure (3.1),

$$Z_N = d_N\ \det_{i,j}\left[\left\langle z^{i-1}\ \middle|\ \bar{z}^{j-1}\right\rangle\right] \tag{3.2}$$

where d_N is an irrelevant numerical constant and we have defined the inner product

$$\left\langle F(z)\ \middle|\ G(\bar{z})\right\rangle = \oint_{z=0} d\mu(z)\ F(z)\,G(\bar{z}) \tag{3.3}$$

on the space $\mathbb{C}[z]\otimes\mathbb{C}[\bar{z}]$ of Laurent polynomials on the unit circle. The construction of orthogonal polynomials to evaluate inner products on the circle such as (3.3) with real, positive definite measure $d\mu$ was discussed long ago in the mathematics literature [21] and subsequently in the physics literature [22]. Some aspects of generic unitary matrix models are also dealt with in [23, 24]. In the present case, the measure (3.1) is complex-valued, and we must deal accordingly with defining the appropriate system of orthogonal polynomials.

These are the monic polynomials $(\Phi_n(z), \Lambda_m(z))$ of order (n, m) which form a complete set in the space $\mathbb{C}[z]\otimes\mathbb{C}[\bar{z}]$ and which are bi-orthogonal in the measure (3.1),

$$\left\langle \Phi_n(z)\ \middle|\ \Lambda_m(z)\right\rangle = h_n\delta_{nm} \tag{3.4}$$

where h_n are some constants. They are normalized as

$$\Phi_n(z) = z^n - \sum_{k=0}^{n-1} p_{n,k}\, z^k \qquad , \qquad \Lambda_m(z) = z^{-m} - \sum_{k=0}^{m-1} l_{m,k}\, z^{-k} \tag{3.5}$$

where the coefficients $p_{n,k}$ and $l_{m,k}$ can be formally obtained from the usual Gram-Schmidt orthogonalization procedure applied to the monomials (z^n, z^{-m}) and the inner product (3.3). In fact, by iterating (3.5) we find that the monomials (z^n, z^{-m}) can be expanded as

$$\begin{aligned} z^n &= \Phi_n(z) + \sum_{j=0}^{n} \left(\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_j=0}^{k_{j-1}-1} p_{n,k_1} p_{k_1,k_2} \cdots p_{k_{j-1},k_j}\, \Phi_{k_j}(z) \right) \\ z^{-m} &= \Lambda_m(z) + \sum_{j=0}^{m} \left(\sum_{k_1=0}^{m-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_j=0}^{k_{j-1}-1} l_{m,k_1} l_{k_1,k_2} \cdots l_{k_{j-1},k_j}\, \Lambda_{k_j}(z) \right) \end{aligned} \tag{3.6}$$

It follows that any polynomial in the space $\mathbb{C}[z] \otimes \mathbb{C}[\bar{z}]$ of degree (n, m) can be expressed as a linear combination of $(\Phi_k(z), \Lambda_\ell(z))$ with $k \leq n$ and $\ell \leq m$. From (3.5) it follows they define a change of basis $(z^{k-1}, z^{-\ell+1}) \mapsto (\Phi_{k-1}(z), \Lambda_{\ell-1}(z))$ of the vector space $\mathbb{C}[z] \otimes \mathbb{C}[\bar{z}]$ that leaves the partition function (3.2) invariant and at the same time diagonalizes the inner product (3.3). In particular, from (3.4) we have

$$Z_N = d_N \, \det_{i,j} \left[\left\langle \Phi_{i-1}(z) \,\middle|\, \Lambda_{j-1}(z) \right\rangle \right] = d_N \; h_0^N \prod_{n=1}^{N-1} R_n^{N-n} \tag{3.7}$$

where we have introduced the recursion coefficients

$$R_n = \frac{h_n}{h_{n-1}} \tag{3.8}$$

and $h_0 = \oint_{z=0} d\mu(z)$ is the normalization of the measure (3.1).

The reason why two independent sets of functions corresponding to clockwise and anti-clockwise rotations on the circle are required here is the complexity of the measure (3.1). In the classical case whereby the measure $d\mu$ is real and positive [21], the polynomials $\Phi_n(z)$ and $\Lambda_n(z)$ are related to each other by complex conjugation. The present system of polynomials may be thought of as a deformation of those for positive definite measures on the unit circle. This deformation is essentially encoded in the coefficients $p_{n+1,0}$ and $l_{n+1,0}$. In order for the Gram-Schmidt process to work for the complex measure (3.1), one needs polynomials in both z and $1/z$ and *a priori* these polynomials are unrelated. Alternatively, we can think of this doubling as the usual doubling of degrees of freedom due to the fermionic nature of the original matrix model. As the orthogonal polynomials define a complete bi-orthogonal system of functions, they satisfy the completeness relation

$$\sum_{n=0}^{\infty} \frac{\Phi_n(z)\, \Lambda_n(z)}{h_n} = \delta^{(\mu)}(z) = z^{N+1} \, \mathrm{e}^{-NV(z)}\, \delta(z) \tag{3.9}$$

on the vector space $\mathbb{C}[z] \otimes \mathbb{C}[\bar{z}]$, where the Dirac delta-function corresponding to the measure $d\mu$ is defined so that

$$\oint_{z=0} d\mu(z)\ F(z)\ \delta^{(\mu)}(z - z') = F(z') \tag{3.10}$$

for any function $F(z)$ on the unit circle. The relation (3.9) is understood as an analytical continuation (to the distribution space that is the completion of $\mathbb{C}[z] \otimes \mathbb{C}[\bar{z}]$) which can be thought of as a representation of the potential $V(\bar{\psi}\psi)$ in terms of the orthogonal polynomials.

The problem of evaluating the partition function of the fermionic matrix model therefore boils down to evaluating the coefficients R_n appearing in (3.7). For this, we will first derive a few properties of the system of orthogonal polynomials introduced above. Completeness implies the relations

$$z\,\Phi_n(z) = \Phi_{n+1}(z) + \sum_{k=0}^{n} \mathcal{P}_k^{(n)}\,\Phi_k(z) \qquad , \qquad \frac{1}{z}\,\Lambda_m(z) = \Lambda_{m+1}(z) + \sum_{k=0}^{m} \mathcal{L}_k^{(m)}\,\Lambda_k(z) \tag{3.11}$$

The coefficients $\mathcal{P}_k^{(n)}$ and $\mathcal{L}_k^{(m)}$ completely determine the form of the orthogonal polynomials and hence the partition function. To see this, we use (3.5) and (3.11) to write

$$\begin{aligned}
\Phi_n(z) &= z\,\Phi_{n-1}(z) - \sum_{k=0}^{n-1} \mathcal{P}_k^{(n-1)}\,\Phi_k(z) \\
&= z^n - \mathcal{P}_0^{(n)} - \sum_{k=1}^{n-1} \Big(p_{n-1,k-1} + \mathcal{P}_k^{(n-1)}\Big)\, z^k - \sum_{k=0}^{n-2} \mathcal{P}_k^{(n-1)} \sum_{j=0}^{k-1} p_{k,j}\, z^j
\end{aligned} \tag{3.12}$$

Comparing the various polynomial coefficients in (3.12) with those of the definition (3.5), we arrive at the iterative relations

$$p_{n,n-1} = p_{n-1,n-2} + \mathcal{P}_{n-1}^{(n-1)} \tag{3.13}$$

$$p_{n,k} = \mathcal{P}_k^{(n-1)} + p_{n-1,k-1} + \sum_{j=k+1}^{n-1} p_{j,k}\,\mathcal{P}_j^{(n-1)} \qquad , \qquad 1 \le k \le n-2 \tag{3.14}$$

$$p_{n,0} = \mathcal{P}_0^{(n)} + \sum_{j=1}^{n-1} p_{j,0}\,\mathcal{P}_j^{(n-1)} \tag{3.15}$$

The relations (3.13) and (3.15) can be iterated straightforwardly and by induction we have

$$p_{n,n-1} = \sum_{j=0}^{n-1} \mathcal{P}_j^{(j)} \tag{3.16}$$

$$\begin{aligned}
p_{n,0} &= \mathcal{P}_0^{(n)} + \sum_{k=1}^{n-3} \left(\sum_{j_1=1}^{n-1} \sum_{j_2=1}^{j_1-1} \cdots \sum_{j_k=1}^{j_{k-1}-1} \mathcal{P}_{j_1}^{(n-1)} \mathcal{P}_{j_2}^{(j_1-1)} \cdots \mathcal{P}_{j_k}^{(j_{k-1}-1)} \mathcal{P}_0^{(j_k)} \right) \\
&\quad + \mathcal{P}_0^{(1)} \mathcal{P}_1^{(1)} \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{j_1-1} \cdots \sum_{j_{n-2}=1}^{j_{n-3}-1} \mathcal{P}_{j_1}^{(n-1)} \mathcal{P}_{j_2}^{(j_1-1)} \cdots \mathcal{P}_{j_{n-2}}^{(j_{n-3}-1)}
\end{aligned} \tag{3.17}$$

which can be substituted into (3.14) to iteratively determine the remaining $p_{n,k}$. Analogous relations exist between the coefficients $l_{m,k}$ and $\mathcal{L}_k^{(m)}$.

We will also need higher degree versions of the completeness relations (3.11). For this, we define constants $P_{n,k}^{[\ell]}$ by

$$z^\ell \, \Phi_n(z) = \sum_{k=0}^{n+\ell} P_{n,k}^{[\ell]} \; \Phi_k(z) \qquad \text{with} \quad P_{n,n+\ell}^{[\ell]} = 1 \tag{3.18}$$

and use (3.11) to write

$$z^{\ell+1} \, \Phi_n(z) = \sum_{k=0}^{n+\ell+1} P_{n,k}^{[\ell+1]} \, \Phi_k(z) = \sum_{k=0}^{n+\ell} P_{n,k}^{[\ell]} \left(\Phi_{k+1}(z) + \sum_{j=0}^{k} \mathcal{P}_j^{(k)} \, \Phi_j(z) \right) \tag{3.19}$$

Using completeness of the orthogonal polynomials we arrive at a recursive relation for the coefficients $P_{n,k}^{[\ell]}$,

$$P_{n,k}^{[\ell+1]} = P_{n,k-1}^{[\ell]} + \sum_{j=k}^{n+\ell} P_{n,j}^{[\ell]} \; \mathcal{P}_k^{(j)} \qquad , \qquad 0 \leq k \leq n+\ell+1 \tag{3.20}$$

with $P_{n,-1}^{[\ell]} \equiv 0$, $\mathcal{P}_k^{(n)} \equiv 0$ for $k > n$, and the initial condition $P_{n,k}^{[0]} = \delta_{nk}$.

3.2 Recursion Relations

We will now determine the coefficients appearing in the completeness relations using the above properties of the orthogonal polynomials. This will lead to a set of recursion equations for the coefficients R_n which thereby completely determines the solution of adjoint fermion one-matrix model. Consider the bi-orthogonality relation

$$\Big\langle \Phi_{n+1}(z) \;\Big|\; \Lambda_m(z) \Big\rangle = 0 \qquad , \qquad 0 \leq m \leq n \tag{3.21}$$

Using the completeness relations (3.11) we can write (3.21) as

$$\Big\langle z \, \Phi_n(z) \;\Big|\; \Lambda_m(z) \Big\rangle = \mathcal{P}_m^{(n)} h_m \tag{3.22}$$

The inner product in (3.22) can be evaluated by grouping the factor of z with the polynomial $\Lambda_m(z)$ with the result

$$\Big\langle z \, \Phi_n(z) \;\Big|\; \Lambda_m(z) \Big\rangle = \Big\langle \Phi_n(z) \;\Big|\; \Lambda_{m-1}(z) + O\left(z^{-m+2}\right) - l_{m,0} \, z \Big\rangle = -l_{m,0} \Big\langle z \, \Phi_n(z) \;\Big|\; 1 \Big\rangle \tag{3.23}$$

where we have used the fact that, aside from the term $l_{m,0} \, z$ which comes from the constant part of the Λ polynomials in (3.5), the function $z \, \Lambda_m(z)$ is a sum of polynomials of degree $\leq m-1$. The final inner product in (3.23) can be evaluated in a straightforward fashion using the relation

$$\Big\langle z \, \Phi_n(z) \;\Big|\; \Lambda_{n+1}(z) \Big\rangle = h_{n+1} \tag{3.24}$$

which follows from (3.11). Again, by grouping the factor of z with $\Lambda_{n+1}(z)$ one finds

$$\Big\langle z\,\Phi_n(z)\;\Big|\;1\Big\rangle = \frac{h_n - h_{n+1}}{l_{n+1,0}} \tag{3.25}$$

Substituting (3.25) and (3.23) into (3.22), we find a concise relation between the constant terms of the Φ polynomials and the completeness coefficients $\mathcal{P}_m^{(n)}$,

$$\begin{aligned}\mathcal{P}_m^{(n)} &= \frac{l_{m,0}}{l_{n+1,0}}\,\frac{h_{n+1}-h_n}{h_m} \\ &= \begin{cases} Q_m\Big(R_{n+1}-1\Big)\displaystyle\prod_{k=m+1}^{n} Q_k R_k & \text{for}\;\; 0\le m<n \\ Q_n(R_{n+1}-1) & \text{for}\;\; m=n \end{cases}\end{aligned} \tag{3.26}$$

where we have defined

$$Q_n = \frac{l_{n,0}}{l_{n+1,0}} \tag{3.27}$$

The completeness relation (3.11) for the Φ polynomials can therefore be written as

$$z\,\Phi_n(z) = \Phi_{n+1}(z) + \frac{h_{n+1}-h_n}{l_{n+1,0}}\sum_{k=0}^{n}\frac{l_{k,0}}{h_k}\,\Phi_k(z) \tag{3.28}$$

Subtracting this relation from itself under the shift of index $n \to n-1$ leads to the three-term recursion relation for the Φ polynomials

$$\Phi_{n+1}(z) = z\,\Phi_n(z) - Q_n\,\frac{R_{n+1}-1}{R_n-1}\Big(\Phi_n(z) - z\,R_n\,\Phi_{n-1}(z)\Big) \tag{3.29}$$

Analogous relations for the Λ polynomials may also be derived. Equating the constant terms on both sides of (3.29) yields

$$\frac{p_{n+1,0}}{p_{n,0}} = -Q_n\,\frac{R_{n+1}-1}{R_n-1} \tag{3.30}$$

so that

$$\begin{aligned}\mathcal{L}_m^{(n)} &= \frac{p_{m,0}}{p_{n+1,0}}\,\frac{h_{n+1}-h_n}{h_m} \\ &= \begin{cases} (-1)^{n-m}\,\dfrac{1-R_m}{Q_m}\,\dfrac{R_{n+1}-1}{R_{m+1}-1}\displaystyle\prod_{k=m+1}^{n}\frac{R_k}{Q_k}\,\frac{R_k-1}{R_{k+1}-1} & \text{for}\;\; 0\le m<n \\ \dfrac{1-R_n}{Q_n} & \text{for}\;\; m=n \end{cases}\end{aligned} \tag{3.31}$$

Substituting (3.31) into the completeness relation (3.11) for the Λ polynomials and subtracting the resulting expression from itself under the index shift $n \to n-1$, we arrive at the three-term recursion relation

$$\Lambda_n(z) = \frac{1}{z}\,R_n\,\Lambda_{n-1}(z) - Q_n\left(\Lambda_{n+1}(z) - \frac{1}{z}\,\Lambda_n(z)\right) \tag{3.32}$$

The recursion relations (3.29) and (3.32) have been derived without any reference to the particular details of the model, i.e. they hold independently of the precise form of the potential V of the fermionic matrix model. Consequently, these relations are merely 'kinematical' in origin. In order to solve for the dynamics of the matrix model we need more information. From a more pragmatic point of view, in order to solve for the partition function we need to find a solution for the coefficients R_n, but the recursion relations also involve the quantities Q_n. More information is needed to link these two objects. As we will see, one only needs two 'dynamical' relations to obtain a closed set of recursion relations for any polynomial potential. The first one is given from (3.4) and (3.5) as

$$\Big\langle \Phi_n(z) \,\Big|\, \Lambda'_{n-1}(z) \Big\rangle = (1-n)h_n \tag{3.33}$$

Integrating by parts on the left-hand side of (3.33) gives

$$(N+1)\Big\langle \Phi_n(z) \,\Big|\, \tfrac{1}{z}\,\Lambda_{n-1}(z) \Big\rangle - N\Big\langle V'(z)\,\Phi_n(z) \,\Big|\, \Lambda_{n-1}(z) \Big\rangle - \Big\langle \Phi'_n(z) \,\Big|\, \Lambda_{n-1}(z) \Big\rangle = (1-n)h_n \tag{3.34}$$

Using the completeness relation (3.11) for the Λ polynomials and the fact that $\Phi'_n(z) = n\Phi_{n-1}(z) + O(z^{n-2})$, we arrive at

$$N\Big\langle V'(z)\,\Phi_n(z) \,\Big|\, \Lambda_{n-1}(z) \Big\rangle = (N+n)h_n - nh_{n-1} \tag{3.35}$$

For a polynomial potential (1.2) of degree K, we may use (3.18) to write (3.35) as an equation for the recursion coefficients R_n,

$$R_n = \frac{n}{n+N} + \frac{N}{n+N} \sum_{k=1}^{K-1} g_{k+1}\, P^{[k]}_{n,n-1} \tag{3.36}$$

Closely related to the first one, the second 'dynamical' relation is given by the bi-orthogonality relation

$$\Big\langle \Phi_n(z) \,\Big|\, \Lambda'_{n-2}(z) \Big\rangle = 0 \tag{3.37}$$

Integrating this by parts gives

$$N\Big\langle V'(z)\,\Phi_n(z) \,\Big|\, \Lambda_{n-2}(z) \Big\rangle = -\Big\langle \Phi'_n(z) \,\Big|\, \Lambda_{n-2}(z) \Big\rangle \tag{3.38}$$

The right-hand side of (3.38) can be evaluated using (3.5), (3.6) and orthogonality, and keeping only those terms in the Φ polynomials which have a non-vanishing overlap with $\Lambda_{n-2}(z)$ in (3.38). This gives

$$\begin{aligned} \Big\langle \Phi'_n(z) \,\Big|\, \Lambda_{n-2}(z) \Big\rangle &= \Big\langle n\,z^{n-1} - (n-1)p_{n,n-1}\,z^{n-2} + O\left(z^{n-3}\right) \,\Big|\, \Lambda_{n-2}(z) \Big\rangle \\ &= \Big[(n-1)p_{n,n-1} - np_{n-1,n-2}\Big] h_{n-2} \end{aligned} \tag{3.39}$$

By defining

$$P_n = \frac{p_{n,n-1}}{N} \tag{3.40}$$

we thus find that the second 'dynamical' relation reads

$$nP_{n-1} - (n-1)P_n = \sum_{k=1}^{K-1} g_{k+1}\, P^{[k]}_{n,n-2} \tag{3.41}$$

The coefficients P_n can be related to the coefficients Q_n and R_n by using (3.16) and (3.26) to get

$$N(P_{n+1} - P_n) = Q_n(R_{n+1} - 1) \tag{3.42}$$

To summarize the results of this subsection, we note that the right-hand sides of (3.36) and (3.41) can be evaluated, using the recursion formula (3.20), in terms of the completeness coefficients $\mathcal{P}^{(n)}_m$. Using (3.26), it follows that, for any polynomial potential, (3.36), (3.41) and (3.42) form a closed set of recursion relations involving only the coefficients R_n, Q_n and P_n. An explicit system of fermionic orthogonal polynomials is constructed in appendix B.

3.3 Free Energy

The final quantity we need for the solution of the fermionic matrix model is the normalization h_0 of the measure (3.1). This integral can be evaluated explicitly to give

$$h_0 = \oint\limits_{z=0} dz\, \frac{\mathrm{e}^{NV(z)}}{z^{N+1}} = \frac{2\pi i}{N!}\, \frac{\partial^N}{\partial z^N}\, \mathrm{e}^{NV(z)} \Big|_{z=0} \tag{3.43}$$

We substitute these quantities into (3.7) and normalize by the Gaussian partition function $Z_N^{\mathrm{Gauss}} = (-1)^{[N]_2}\,(Ng_1)^{N^2}$ for which $V(z) = g_1 z$, $R_n = n/(n+N)$ and $h_0 = 2\pi i (Ng_1)^N/N!$. For a polynomial potential, we may evaluate (3.43) explicitly using (2.11), and using the multinomial theorem we thus find that the free energy $F_N = \frac{1}{N^2}\log\frac{Z_N}{Z_N^{\mathrm{Gauss}}}$ is given by

$$\begin{aligned} F_N \;=\; & \sum_{n=1}^{\infty} (-1)^n\,(n-1)! \sum_{0\le k_0^{(2)},\ldots,k_{[N]_2}^{(2)}\le n} \cdots \sum_{0\le k_0^{(K)},\ldots,k_{[N]_K}^{(K)}\le n} \frac{\delta\left(\sum_{\ell,m} k_\ell^{(m)} - n\right)}{\prod_{\ell,m} k_\ell^{(m)}!} \\ & \times \prod_{j=2}^{K} \prod_{\ell_j=1}^{[N]_j} \left[\frac{N!}{(N-\sum_i i\ell_i)!\,\ell_2!\cdots\ell_K!} \prod_{m=2}^{K} \left(\frac{g_m}{mN^{m-1}(g_1)^m} \right)^{\ell_m} \right]^{k_{\ell_j}^{(j)}} \\ & + \frac{1}{N} \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \log \frac{(n+N)R_n}{n} \end{aligned} \tag{3.44}$$

where the coefficients R_n are determined from (3.36).

Generally, in the large-N limit, the variable $x \equiv n/N$ becomes a continuous parameter in the interval $[0,1]$. We assume that in this limit the recursion coefficients $\mathcal{R}_n$ (any of the quantities R_n, Q_n or P_n), determined as the solutions of the recursion equations derived in the previous subsection in the large-N limit, become continuous functions $\mathcal{R}(x)$ of

$x \in [0,1]$. This is justified by the form of the recursion relations which suggest the replacement

$$\mathcal{R}_n \to \mathcal{R}(\tfrac{n}{N}) = \mathcal{R}(x) \tag{3.45}$$

It follows that shifts in the index n are related to sub-leading contributions in the large N limit,

$$\mathcal{R}_{n+a} \to \mathcal{R}(x) + \frac{a}{N} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{a}{N}\right)^{k-1} \frac{\partial^k \mathcal{R}(x)}{\partial x^k} \tag{3.46}$$

and the second, finite sum in (3.44) can be approximated by a simple one-dimensional integral over $x \in [0,1]$ at $N = \infty$. To deal with the first term in (3.44), we need to determine the large N limit of the normalization constant h_0. For this, we note that there is nothing particularly special about the choice of integration contour in (3.43), as it was merely introduced in (2.19) as a means of evaluating the derivatives in (2.9). In particular, since its integrand is an analytic function, we can deform the contour arbitrarily in the complex plane, and in particular to one out at infinity. In the large N limit, the one-dimensional integral h_0 may then be evaluated using the saddle-point approximation. In this way we find that the genus 0 free energy is given by

$$F_0 \equiv \lim_{N\to\infty} F_N = f_0 + \int_0^1 dx\ (1-x) \log \frac{(1+x)R(x)}{x} \tag{3.47}$$

where

$$f_0 = V(\zeta) - \log g_1 \zeta \tag{3.48}$$

with ζ the solution of the order K algebraic saddle-point equation

$$\zeta\, V'(\zeta) = 1 \tag{3.49}$$

for the branch which has the perturbative Gaussian limit $\zeta|_{g_2=\ldots=g_K=0} = 1/g_1$ and for which the function (3.48) is real-valued.

4 Topological Expansion

In this section we will solve for the continuum limit of the discretized random surface theory represented by the adjoint fermion one-matrix model. We will consider two limits of the matrix model. The first one is the naive large-N limit which captures the leading critical behaviour and gives a non-perturbative solution to the problem of counting fermionic ribbon graphs of spherical topology. The second one is the double-scaling limit in which an appropriate coupling constant is tuned in the limit $N \to \infty$ in order to obtain contributions from all orders of the $\frac{1}{N}$ expansion of the free energy and which yields the complete topological expansion of the fermionic matrix model.

4.1 Critical Behaviour

We will start by making some general remarks concerning the large N limit of the fermionic matrix model. From (3.47) we see that there are two contributions to the $N = \infty$ free energy. These two quantities represent very different phases of the system. The first one is given by (3.48) which is determined by the saddle-point equation (3.49). This equation is immediately recognized as the critical equation describing the continuum limit of an abstract branched polymer [5, 7]. In fact, the explicit derivative expression for the measure normalization (3.43) coincides exactly with the partition function of the N dimensional fermionic *vector* model with the same polynomial potential V [14]. The latter quantity is related to the generating function for branched polymer networks by a simple mapping of its Feynman diagrams and an analytical continuation $N \mapsto -\frac{N}{2}$ of the fermionic vector dimension. The analytic reason for the appearance of branched polymer behaviour in the fermionic matrix model is that the normalization of its measure is not of sub-leading order in N. From a graphical point of view, its appearance is clear from the analysis of section 2.2. The entropy factors associated with the number of Feynman graphs is drastically reduced in the fermionic case because of the cancellations which occur between twisted diagrams. There is a class of diagrams with no twists in the fermionic matrix model, and these have a tree-like growth in the large N limit. The reduced entropy of the remaining twisted graphs is then comparable to that of the graphs which produce a polymer-like behaviour. This feature is unique to fermionic matrices, and it is the property which enables the construction of a model of branched polymers using supersymmetric matrix models [13] via the coupling of the fermionic matrix model to an ordinary, complex matrix model which has the effect of cancelling the set of twisted diagrams. In fact, it is precisely the twisting mechanism described in section 2.2 that enables one to isolate graphs with tree-like growth in the fermionic case. The untwisted diagrams may then be mapped onto branched polymers similarly to the case of the cactus diagrams which appear in supersymmetric matrix models [13].

The twisted graphs are associated with the second contribution in (3.47) and, as we will demonstrate, they lead to the usual surface effects in the theory. Thus the orthogonal polynomial formalism that we have developed naturally splits the generating function into a piece corresponding to the tree-like graphs and a piece corresponding to the twisted diagrams. The continuum limit of the latter ensemble of graphs describes a surface theory and describes an effect which is of order N^2, while the former ensemble which has a polymer growth produces an effect of order N. One typical feature of the polymer generating function is that it becomes non-analytic in the large N limit at the Gaussian point $g_k = 0$, $k \geq 2$. However, this effect is of sub-leading order in N, so that a large N analysis with arbitrary coupling constants will miss the branched polymer phase transition. There is a "barrier" at the Gaussian point which separates the surface-like continuum limit from the branched polymer behaviour. From a geometrical point of view, the fermionic ma-

trix model probes an intermediate lattice phase where the number of Feynman diagrams contributing to a given process is greater than that expected of a vector model but less than that of a matrix model. Thus the matrix integral generates a random surface theory which contains an "internal" branched polymer phase.

This decrease in the number of graphs as compared to the usual matrix models will be responsible for the Borel summability of the all genus expansion of the free energy of the fermionic matrix model that we will prove in this section. As discussed above, the essential critical behaviour of the fermionic matrix model is encoded in the large-N limit of the recursion coefficients R_n. If we introduce the quantities τ_n defined by

$$h_n = \frac{\tau_{n+1}}{\tau_n} \tag{4.1}$$

with $\tau_0 = d_N$ as in (3.2), then the recursion coefficients are given by $R_n = \tau_{n+1}\,\tau_{n-1}/\tau_n^2$ and the free energy by $F_N = \mathcal{F}_N$, where $\mathcal{F}_n = \log(\tau_n/\tau_n^{\rm Gauss})$. Replacing sequences by functions of $x \in [0,1]$ as prescribed by (3.46), we then have to leading order in N,

$$\log\frac{(1+x)R(x)}{x} = \frac{\partial^2\mathcal{F}(x)}{\partial x^2} \qquad , \qquad F_0 = \mathcal{F}(1) \tag{4.2}$$

The relations (4.2) show that the function $R(x)$ is related to the specific heat u_0 of the matrix model, and also that any singularity in the free energy will occur at the boundary $x = 1$ of the domain of $R(x)$. These facts will enable us to construct the topological expansion of the fermionic matrix model from the recursion coefficients R_n of the orthogonal polynomials. We remark that it is possible to demonstrate from the constrained sum over $SU(N)$ group characters of section 2.3 that the partition function is real and has a definite value for any N. Combining this with our knowledge from large N loop equation calculations, wherein the system goes to a definite limit, one can argue that there is a definitive $\frac{1}{N}$ expansion of the free energy. The phase transition at large N can thereby be understood as a sort of percolation transition in the Young tableaux, whereby the system evolves into a phase in which the number of filled $SU(N)$ Young tableaux grows asymptotically.

4.2 Main Recursion Relation

We will now begin quantifying the above discussion. For definiteness, in the remainder of this section we will deal primarily with the simplest non-Gaussian model which is given by the quadratic potential

$$V(z) = z + \frac{g}{2}\,z^2 \tag{4.3}$$

although, as we will discuss, the results we obtain are universal. The potential (4.3) is difficult to treat within the loop equation approach because the endpoints of the support of the corresponding spectral density are located asymmetrically in the complex plane [10]. Indeed, one power of the technique that we develop is that it is insensitive to the

parity symmetry (or lack thereof) of the potential. This will account for the universality properties of the solution that we shall find. In this subsection we will derive the main recursion relation corresponding to the potential (4.3) that will be used to construct the topological expansion.

To solve the recursion relations (3.36), (3.41) and (3.42) corresponding to (4.3), we first use (3.20) and (3.26) to determine the coefficients

$$\begin{aligned} P^{[1]}_{n,n-1} &= \mathcal{P}^{(n)}_{n-1} = Q_{n-1}Q_nR_n(R_{n+1}-1) \\ P^{[1]}_{n,n-2} &= \mathcal{P}^{(n)}_{n-2} = Q_{n-2}Q_{n-1}Q_nR_{n-1}R_n(R_{n+1}-1) \end{aligned} \tag{4.4}$$

This leads to the closed set of recursion relations

$$(N+n)R_n - n = g\,N\,Q_{n-1}Q_nR_n(R_{n+1}-1) \tag{4.5}$$

$$nP_{n-1} - (n-1)P_n = g\,Q_{n-2}Q_{n-1}Q_nR_{n-1}R_n(R_{n+1}-1) \tag{4.6}$$

$$N(P_{n+1}-P_n) = Q_n(R_{n+1}-1) \tag{4.7}$$

From the relations (4.6) and (4.7) one can solve for the coefficients P_n as

$$N\,P_n = g\,N\,Q_{n-2}Q_{n-1}Q_nR_{n-1}R_n(R_{n+1}-1) + n\,Q_{n-1}(R_n-1) \tag{4.8}$$

Substituting (4.8) into (4.7) and using (4.5) to reduce the degree of the Q_n's in the resulting equation yields

$$\begin{aligned} 0 &= Q_{n-1}R_n\Big[(N+n+1)R_{n+1}-n-1\Big] - Q_{n-2}R_{n-1}\Big[(N+n)R_n-n\Big] \\ &\quad + n\,Q_n(R_{n+1}-1) - n\,Q_{n-1}(R_n-1) \end{aligned} \tag{4.9}$$

We now use (4.5) to define the quantity

$$\Omega_n\{R\} \equiv Q_nQ_{n-1} = \frac{1}{g}\,\frac{(N+n)R_n-n}{N\,R_n(R_{n+1}-1)} \tag{4.10}$$

and multiply the relation (4.9) through by Q_n. Using (4.6) along with (4.8), we can then solve for Q_n in terms of the R_n's alone as

$$Q_n\{R\}^2 = \Omega_n\{R\}^2\,\frac{n(R_n-1) - R_n\Big[(N+n+1)R_{n+1}-n-1\Big]}{n\,\Omega_n\{R\}(R_{n+1}-1) - \Omega_{n-1}\{R\}\,R_{n-1}\Big[(N+n)R_n-n\Big]} \tag{4.11}$$

We may therefore write down a recursion relation involving *only* the coefficients R_n in the form

$$Q_n\{R\}^2\,Q_{n-1}\{R\}^2 = \Omega_n\{R\}^2 \tag{4.12}$$

where the function $Q_n\{R\}^2$ is given by (4.11) and $\Omega_n\{R\}$ by (4.10). Notice that the coupling constant g completely cancels out in this final equation which is an involved non-linear recursion relation for the R_n's solely in terms of n and N. The dependence of R_n on g comes about purely as a boundary condition and can only be recovered after solving for the dynamics of R_n. However, we will see that the entire topological expansion can be constructed without ever knowing this dependence explicitly. All information about the continuum limit will come from the details of the main recursion relation (4.12).

4.3 Planar Limit

The spherical continuum limit of the matrix model is found by taking the limit $N \to \infty$. Let us first describe the polymer contribution f_0 in (3.47). For the potential (4.3), the solution of the quadratic saddle-point equation (3.49) which is regular at $g = 0$ is given by

$$\zeta = \frac{1}{2g}\left(\sqrt{1+4g}-1\right) \tag{4.13}$$

so that the corresponding fermionic vector free energy (3.48) is

$$f_0(g) = \frac{1}{2} + \frac{1}{4g}\left(\sqrt{1+4g}-1\right) - \log\frac{1}{2g}\left(\sqrt{1+4g}-1\right) \tag{4.14}$$

There is a phase transition at $g = g^{(0)} = -\frac{1}{4}$ below which the free energy becomes complex-valued. Near this critical point, (4.14) behaves as $f_0(g) \sim (g - g^{(0)})^{3/2}$, which identifies the corresponding string susceptibility exponent as $\gamma_{\rm str}^{(0)} = +\frac{1}{2}$. The phase transition is therefore associated with the evolution of the system into a pure branched polymer phase of two-dimensional quantum gravity. In fact, the function (4.14) is related to the usual bosonic vector model free energy $\mathcal{F}_{\rm vec}$, which is the generating function for pure, connected branched polymer graphs, by $f_0 = 1 - 2\mathcal{F}_{\rm vec}$ [14]. The N dimensional Euclidean radial coordinate r of the bosonic vector model is related to the saddle-point (4.13) by $r^2 = \frac{1}{2}\,\zeta$. One can also compute the one-loop fluctuations around the saddle-point (4.13) and show that the resulting free energy is logarithmically divergent at $g = 0$ [14]. Therefore, in the regime $g > 0$ there is no polymer behaviour and one may imagine a "barrier" in the large N limit of the theory at the Gaussian point $g = 0$. It is further possible to carry out a double scaling expansion of the fermionic vector model partition function (3.43) in the limit $N \to \infty, g \to g^{(0)}$ with the parameter $N(g - g^{(0)})^{3/2}$ held fixed. The resulting expression is an alternating, Borel summable series which can be expressed as a Bessel function [14].

We now consider the large N behaviour associated with the second term in the free energy (3.47). Replacing the discrete R_n coefficients with the large-N continuous function $R(x)$ as prescribed by (3.46), after some algebra we obtain from the recursion relation (4.12), at leading order in N, the first order non-linear ordinary differential equation

$$\frac{dR}{dx} = \frac{R\left(R-1\right)^2\left(x + 2x\,R - 3(1+x)\,R^2\right)}{x^2 - 2x^2\,R + 4x(1+x)\,R^2 - 2(1+x)(2+3x)\,R^3 + 3(1+x)^2\,R^4} \tag{4.15}$$

The dependence of R on the coupling constant g arises only from the constant of integration of (4.15). It is known from the general theory of ordinary differential equations [25] that the solutions $R(x)$ to equations such as (4.15) possess algebraic non-analytic behaviour. Since dR/dx in (4.15) is given as the quotient of a quintic polynomial in R by a quartic one, it follows that the function $R(x)$ is finite for all finite values of x [25]. The only singularities which can arise from this differential equation occur when the denominator on the right-hand side vanishes and the numerator is non-zero. In light of the

discussion of the previous subsection, we will demand that these singular points occur at the boundary $x = 1$ of the domain of the function $R(x)$. At $x = 1$, the denominator of the right-hand side of (4.15) has four distinct zeros R_c determined as the solutions of the quartic equation

$$1 - 2R_c + 8R_c^2 - 20R_c^3 + 12R_c^4 = 0 \tag{4.16}$$

Denoting the corresponding critical value of the coupling constant g by g_c, we make an ansatz for the form of the function $R(x; g)$ near the critical point,

$$R(x; g) = R_c + a \cdot (g_c - gx)^{-\gamma_{\rm str}} + \dots \tag{4.17}$$

where a is some constant and the ellipsis denotes terms which are less singular at the critical point. This ansatz ensures that for $g \sim g_c$, the singularities occur at $x = 1$, or alternatively that at $x = 1$ the free energy $\mathcal{F}(x)$ becomes non-analytic at $g = g_c$. Substituting (4.17) into the differential equation (4.15), we find for each branch R_c of the equation (4.16) the leading behaviour

$$\frac{dR}{dx} \sim \frac{1}{R_c - R(x)} \tag{4.18}$$

Comparing with (4.17) fixes the exponent $\gamma_{\rm str} = -\frac{1}{2}$, and so the critical point $g = g_c$ of the fermionic matrix model describes a continuum limit that lies in the same universality class as pure two-dimensional quantum gravity in the planar limit. Note that this criticality argument requires no knowledge of the precise value of the critical coupling g_c. The same will be true of the double scaling limit which will be analysed in the next subsection. It is possible to show using loop equations [9] that $g_c > 0$. Thus the continuum surface behaviour occurs on the opposite side of the Gaussian point relative to the branched polymer phase transition, and the two phases cannot be connected together in a smooth way.

We conclude our analysis of the planar limit of the adjoint fermion one-matrix model by briefly describing how it extends to more general potentials. For example, consider a potential of the generic form $V(z) = z + \frac{g}{K}\, z^K$, $K \geq 2$. The recursion equations (3.36) and (3.41) in this case require knowledge of the coefficients $P^{[K]}_{n,n-1}$ and $P^{[K]}_{n,n-2}$ determined from (3.20) and (3.26). It is straightforward to see from these relations that $P^{[K]}_{n,n-1}$ is of degree K in both the Q_n's and the R_n's, while $P^{[K]}_{n,n-2}$ is of degree $K + 1$. In the large N limit, the recursion equations will therefore assume the form

$$(1 + x)R - x = g\, Q^K\, W_1[R] \tag{4.19}$$

$$P - x\, \frac{\partial P}{\partial x} = g\, Q^{K+1}\, W_2[R] \tag{4.20}$$

$$\frac{\partial P}{\partial x} = Q\Big(R - 1\Big) \tag{4.21}$$

where $W_1[R]$ and $W_2[R]$ are polynomials of degree K and $K + 1$ in the function R, respectively, which each contain the factor $R - 1$. Using (4.19) we can solve for Q^K and

write (4.20) as an equation determining the function Q. Comparing with the solution for Q obtained using (4.21), the relation (4.20) then becomes a first order, linear, homogeneous differential equation for the function P. Substituting the resulting solution for P back into (4.19) yields an integral equation for the function R which, upon differentiation, can be transformed into a first order non-linear ordinary differential equation of the form

$$\frac{dR}{dx} = \frac{J_2[R;x]}{J_1[R;x]} \tag{4.22}$$

where $J_1[R;x]$ and $J_2[R;x]$ are polynomial functions of their arguments of respective degrees $2K$ and $2K+1$ in R which are independent of the coupling constant g. Arguing as we did above, the differential equation (4.22) will admit critical behaviour for the function R of the square root type, i.e. that of pure gravity. Provided that the zeros of the function $J_1[R;x]$ are non-degenerate at $x=1$, this will be the only type of critical behaviour that the model admits. It is of course natural that pure gravity exists for a generic matrix potential. Multicritical points require more complicated potentials with two or more coupling constants and an appropriate fine-tuning of them. In this case the differential equations derived analogously to those above become highly non-linear and quite involved. But the general picture will be the same, namely the function R will be determined by some differential equation whose polynomial coefficients (in R and x) can be tuned to obtain higher order critical points. In this way we can recover, in the planar limit, the standard universality classes of conformal matter coupled to two-dimensional quantum gravity [4].

4.4 Double Scaling Limit

In this subsection we will study the all genus expansion of the specific heat

$$u(g,N) = \sum_{h=0}^{\infty} N^{-2h}\, u_h(g) \qquad , \qquad u_h(g) = F_h''(g) \tag{4.23}$$

where $F_h(g)$ is the genus h contribution to the free energy $F_N(g)$ and the genus h susceptibility is a homogeneous function of degree $-2h$, $u_h(a\cdot g) = a^{-2h}\cdot u_h(g)$. From the analysis of the previous subsection, we know that the leading singular behaviour of (4.23) at genus zero is $u_0(g) \sim \sqrt{g-g_c}$. In order to keep the partition function $Z_0(g) \sim \mathrm{e}^{N^2 F_0(g)}$ finite in the large N limit, we should therefore perform the double scaling limit $g \to g_c, N \to \infty$ by keeping fixed the parameter

$$\kappa = N^{4/5}\,(g_c - g) \tag{4.24}$$

We now define a scaling function $u(\kappa)$ which captures the contributions to the string susceptibility from all genera in the double scaling limit $g \to g_c, N \to \infty$ with the variable (4.24) fixed. For this, we introduce the large-N expansion parameter $\epsilon = \frac{1}{N}$ and write the total susceptibility (4.23) in the vicinity of the critical point as

$$u(g,N) = (\epsilon^2)^{4/5}\, a_0 + \epsilon^2\, u(\kappa) + (\epsilon^2)^{6/5}\, a_1\, \kappa + O\left((\epsilon^2)^{7/5}, \kappa^2\right) \tag{4.25}$$

where a_n are constants. By defining the new effective variable

$$\xi = \epsilon^{-4/5}\,(g_c - gx) \tag{4.26}$$

in the large N limit, this scaling function may be determined from the $\xi = \kappa$ limit of a scaling ansatz for the function $R(x)$ in terms of an unknown function $u(\xi)$,

$$R(x) = R_c + \epsilon^{2/5}\,u(\xi) \tag{4.27}$$

where R_c is a solution of (4.16). By substituting (4.27) into (3.47) and changing variables from x to ξ in the integral, we find that, up to irrelevant constants (which may be absorbed by suitable rescaling using the homogeneity of the specific heat) and terms which vanish as $\epsilon \to 0$, the free energy in the double scaling limit is determined as

$$F(\kappa) = f_0 + \int\limits_{\epsilon^{-4/5}\,g_c}^{\kappa} d\xi\ (\kappa - \xi)\,u(\xi) \tag{4.28}$$

It follows that

$$u(\kappa) = F''(\kappa) \tag{4.29}$$

and so the problem of obtaining the topological expansion of the fermionic matrix model is thereby reduced to the task of finding the solution of the main recursion relation (4.12) for the function $R(x)$ in this special limit. Note that the polymer free energy (4.14) contributes only an irrelevant constant in the double scaling limit about the positive-valued critical point g_c.

We now rewrite (4.12) using (3.46) and take the limit of large-N while holding the quantity κ fixed through the relation $g = g_c - \kappa N^{-4/5}$. We then substitute in the scaling ansatz (4.27) and rewrite derivatives according to the change of variables (4.26), i.e. $\partial_x \sim -\epsilon^{-4/5}\,\partial_\xi$. After some algebra, we find, as the coefficient of the leading order $\epsilon^{4/5}$ term, a non-linear differential equation for the specific heat $u(\xi)$,

$$\begin{aligned} 0 \;=\;& 6g_c\Big(3 - 22R_c + 80R_c^2 - 192R_c^3 + 376R_c^4 - 632R_c^5 + 752R_c^6 - 512R_c^7 + 144R_c^8\Big) \\ & \times u(\xi)\,u'(\xi) + \;R_c\Big(1 - 5R_c + 10R_c^2 - 10R_c^3 + 4R_c^4\Big) \\ & \times \Big[6R_c\Big(R_c - 1\Big)^2\Big(6R_c^2 - 2R_c - 1\Big) - g_c^3\;\Big(1 - 2R_c + 20R_c^2 - 56R_c^3 + 36R_c^4\Big)\;u'''(\xi)\Big] \end{aligned} \tag{4.30}$$

We now integrate (4.30) up once and use an inessential shift of the independent variable ξ to eliminate the constant term. After applying (4.16) and a rescaling using the homogeneity of the function $u(\xi)$ (which preserves the relationship (4.29)), we arrive at the parameter-free equation

$$u''(\xi) + u(\xi)^2 = \xi \tag{4.31}$$

The non-linear differential equation (4.31) governs the behaviour of the partition function of the fermionic random surface model to all orders in the genus expansion, with the

parameter $\xi^{5/4}$ identified as the renormalized string coupling constant. We note again the remarkable feature that one never needs to know the precise location of the critical point to arrive at this equation. In the present case the critical coupling constant g_c can be consistently rescaled out of the pertinent equations. This parametric independence is indicative of the universality of the double scaling equation (4.31) for the given class of generic polynomial potentials.

The differential equation (4.31) is known as the Painlevé I equation and it is solved by the first Painlevé transcendent. This equation differs from the versions which usually appear in matrix models [4] by a change in sign of the specific heat $u \to -u$. The boundary conditions which $u(\xi)$ satisfies follow from the analysis of the planar limit of the previous subsection. The genus zero contribution arises in the limit of large, positive ξ where the leading behaviour is $u(\xi)^2 = \xi$. As a consequence, we are led to postulate an asymptotic expansion for $u(\xi)$ of the form

$$u(\xi) = \sqrt{\xi}\,\left(1 + \sum_{k=1}^{\infty} u_k\,\xi^{-5k/2}\right) \tag{4.32}$$

By substituting (4.32) into (4.31), the first few coefficients are easily calculated to be

$$\begin{aligned} u_1 &= \frac{1}{8} \\ u_2 &= -\frac{49}{128} \\ u_3 &= \frac{1225}{256} \\ u_4 &= -\frac{4412401}{32768} \\ u_5 &= \frac{220680075}{32768} \\ u_6 &= -\frac{2207064977649}{4194304} \\ &\vdots \end{aligned} \tag{4.33}$$

and in general they are determined by the recursive equation

$$u_k = -\frac{(5k-6)(5k-4)}{8}\,u_{k-1} - \frac{1}{2}\sum_{n=1}^{k-1} u_k\,u_{k-n} \qquad , \qquad k \geq 2 \tag{4.34}$$

With the normalization of the spherical specific heat taken as $u_0 = 1$, we see from (4.33) that the coefficients of the genus expansion have *precisely* the same absolute numerical values as those obtained for pure two-dimensional quantum gravity, i.e. from the usual Painlevé expansion [4, 5]. In particular, they have the same high rate asymptotic growth in magnitude. However, the most notable feature of the coefficients (4.33) is the fact that they oscillate in sign (In the Hermitian cases, all u_k are negative). This raises the possibility that the asymptotic series solution (4.32) may in fact be Borel summable. If

so, then the free energy would give an unambiguous definition of the genus expansion of pure "fermionic" gravity in powers of $N^{-2} \sim \xi^{-5/2}$. A numerical solution of the equation (4.31) is possible and the result is shown in fig. 1. This confirms numerically that the free energy exists as a well-defined function and is real-valued on the positive real ξ-axis. In the next subsection we shall prove that the coefficients u_k of the asymptotic series (4.32) alternate in sign to arbitrarily large orders of the genus expansion, and, moreover, that there is a well-defined Borel resummation of $u(\xi)$ for $\xi > 0$.

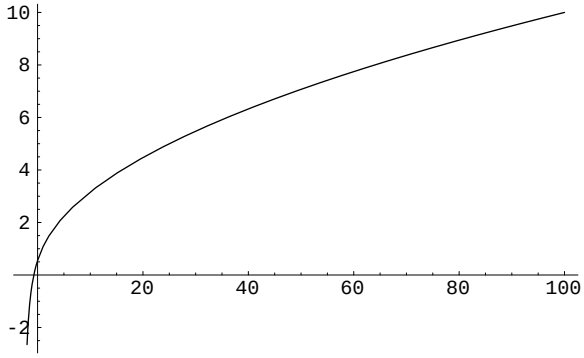


Figure 1: *Plot of the solution $u(\xi)$ versus ξ of the Painlevé equation with the asymptotic boundary condition $u(\xi)^2 = \xi$ for $\xi \to +\infty$. The first double pole singularity (corresponding to a zero of the partition function) occurs on the negative axis at $\xi \simeq -3.1477$.*

4.5 Borel Summability

In this subsection we will argue analytically, along the lines of [4], that the asymptotic series (4.32) defines a unique function $u(\xi)$. For this, we consider the Borel transform

$$B(s) = \sum_{k=1}^{\infty} \frac{u_k}{(\beta k)!}\, s^k \tag{4.35}$$

where the constant β will be self-consistently determined by the condition that this series has a finite radius of convergence. A solution of the original Painlevé equation (4.31) is then given by

$$u(\xi) = \sqrt{\xi}\, \left(1 - \int_0^{\infty} dt\ \mathrm{e}^{-t}\, B\left(t^{\beta}\, \xi^{-5/2} \right) \right) \tag{4.36}$$

The crucial issue now is whether or not the integral in (4.36) actually exists. We will show that there is a contour running through the $\mathrm{Re}\, t > 0$ region of the complex t-plane from $t = 0$ to $t = \infty$ along which the integral transform (4.36) converges. We will thereby argue that the Borel resummation (4.36) of the specific heat is well-defined and real-valued.

The function (4.35) has singularities in the complex s-plane with branch cuts running between them and infinity. If some of these singularities lie on the positive s-axis then

the integral definition (4.36) is ambiguous. We will now extract the large order behaviour of the coefficients u_k and argue that the singularities of the Borel transform $B(s)$ all lie on the negative s-axis. From (4.35) and (4.36) it follows that the coefficients of the asymptotic expansion (4.32) are given by

$$u_k = \frac{1}{2\pi i} \int_0^\infty dt\ \mathrm{e}^{-t} \oint_{s=0} \frac{ds}{s^{k+1}}\ B\left(t^\beta\, s\right) \tag{4.37}$$

The contour of integration in (4.37) can be extended out to infinity around the cuts of the Borel transform in the complex s-plane. In the limit $k \to \infty$, the contour integral in (4.37) is dominated by contributions along the branch cut of $B(s)$ in the complex s-plane which begins at the point $s = s_0$ that is closest to the origin. We then have, for large k,

$$u_k \sim \int_0^\infty dt\ \mathrm{e}^{-t} \int_{s_0/t^\beta}^\infty \frac{ds}{s^{k+1}}\ \mathrm{Disc}\, B\left(t^\beta\, s\right) \sim \int_0^\infty \frac{ds}{s^{k+1}} \int_{(s_0/s)^{1/\beta}}^\infty dt\ \mathrm{e}^{-t}\ \mathrm{Disc}\, B\left(t^\beta\, s\right) \tag{4.38}$$

where we have defined the discontinuity of the Borel transform

$$\mathrm{Disc}\, B(s) = B_+(s) - B_-(s) \qquad , \qquad B_\pm(s) = B(s \pm i0) \tag{4.39}$$

for s a point on its cut.

Going back to (4.36), we see from (4.38) that we should study the functions $u_\pm(\xi)$ corresponding to the Borel transforms on either side of the dominant cut,

$$u_\pm(\xi) = \int_{(s_0\, \xi^{5/2})^{1/\beta}}^\infty dt\ \mathrm{e}^{-t}\, B_\pm\left(t^\beta\, \xi^{-5/2}\right) \tag{4.40}$$

Since each of the functions (4.40) solves the Painlevé equation (4.31), it is convenient to define their sum and difference

$$u_{\mathrm{d}}(\xi) = u_+(\xi) - u_-(\xi) = \mathrm{Disc}\, B(\xi) \qquad , \qquad u_{\mathrm{s}}(\xi) = \tfrac{1}{2}\left(u_+(\xi) + u_-(\xi)\right) \tag{4.41}$$

which on using (4.31) are seen to obey the coupled system of differential equations

$$u_{\mathrm{d}}''(\xi) + 2u_{\mathrm{d}}(\xi)\, u_{\mathrm{s}}(\xi) = 0 \tag{4.42}$$

$$u_{\mathrm{d}}''(\xi) + u_{\mathrm{s}}(\xi)^2 + \frac{1}{4}\, u_{\mathrm{d}}(\xi)^2 = \xi \tag{4.43}$$

These equations can be solved in the WKB approximation. In the planar limit $\xi \to \infty$, we self-consistently assume that $u_{\mathrm{d}}(\xi)$ is of sub-leading order in (4.43) and thereby find, to leading order, the solution $u_{\mathrm{s}}(\xi) \sim \sqrt{\xi}$. By substituting this into (4.42), to leading order we have the first order linear differential equation for the function $u_{\mathrm{d}}(\xi)$,

$$u_{\mathrm{d}}''(\xi) + 2\sqrt{\xi}\, u_{\mathrm{d}}(\xi) = 0 \tag{4.44}$$

which can be solved in terms of Bessel functions as $u_{\rm d}(\xi) = \sqrt{\xi}\, Z_{\frac{2}{5}}\left(\frac{4\sqrt{2}}{5}\,\xi^{5/4}\right)$. The solution for $\xi \to \infty$ is therefore given by

$$\frac{u_{\rm d}(\xi)}{\sqrt{\xi}} \sim \xi^{-5/8}\, \cos\left(\frac{4\sqrt{2}}{5}\,\xi^{5/4} \mp \frac{5\pi}{4} - \frac{\pi}{4}\right) \tag{4.45}$$

We are finally ready to estimate the large order behaviour of the coefficients u_k. For this, it is convenient to change variables in order to bring the asymptotic expansion for $\xi \to \infty$ to $\omega \to 0$ through the definition $\omega = \xi^{-5/4}$. Then, from (4.38)–(4.41) and (4.45), we find as $k \to \infty$,

$$u_k \sim \int \frac{d\omega}{\omega^{2k+1}}\; \omega^{2/5}\, u_{\rm d}\left(\omega^{-4/5}\right) \sim \int \frac{d\omega}{\omega^{2k+1}}\; \sqrt{\omega}\, \cos\left(\frac{4\sqrt{2}}{5\omega} \mp \frac{5\pi}{4} - \frac{\pi}{4}\right) \tag{4.46}$$

For each choice of sign the integral (4.46) gives, up to constant factors, the result

$$u_k \sim \left(-\frac{32}{25}\right)^{-k}\, \Gamma\left(2k - \tfrac{1}{2}\right) \qquad \text{for} \quad k \to \infty \tag{4.47}$$

The asymptotic estimate (4.47) contains a large amount of information. Going back to the Borel transform (4.35), the large order behaviour of u_k fixes $\beta = 2$ and gives

$$B(s) \sim \sum_k \left(-\frac{32}{25}\right)^{-k}\, s^k \tag{4.48}$$

We see therefore that the Borel transform $B(s)$ is well-defined in an open region of the complex s-plane and, moreover, that its first singularity appears on the negative real s-axis. Assuming that all of its singularities are so restricted, we can expect that the asymptotic series (4.32) can be Borel resummed through the integration in (4.36) to define a unique real function $u(\xi)$ on the positive real ξ-axis. The results of a numerical analysis (fig. 1) support these arguments.

From this analysis it follows that the double-scaled partition function of the adjoint fermion one-matrix model leads to a completely well-defined genus expansion of the corresponding random surface theory. The coefficients (4.47) have precisely the same large order behaviour $\Gamma(2k - \frac{1}{2})$ as those of pure two-dimensional quantum gravity. In fact, the topological expansion of the fermionic matrix model is identical term by term to that of the usual Hermitian matrix models. However, the alternating nature of the asymptotic series expansion allows one to define the string susceptibility in an unambiguous way through its Borel resummation (4.36), at least for positive values of the string coupling ξ. In this way we may think of the fermionic matrix model as lending a well-defined version of the generating functions provided by Hermitian matrix models. It gives an analytic solution to the problem of counting fermionic random triangulations on arbitrary genus Riemann surfaces, whose asymptotic expansion is Borel summable but otherwise coincides with the usual Painlevé expansion of two-dimensional quantum gravity.

The usual movable, double pole singularities of the Painlevé I equation appear as well in the present case, but this time they appear for *negative* values of the cosmological constant ξ. The existence of such poles is inconsistent with the loop equations of two-dimensional quantum gravity [26]. Pole-free solutions of Painlevé equations are provided by the triply-truncated Boutroux solutions, but these are complex-valued and do not lead to physically acceptable free energy functions. However, in the present situation there is a unique, well-defined real-valued specific heat on the positive ξ-axis which should have a meaningful analytical continuation to $\xi < 0$. It provides an unambiguous definition of a whole branch of the random surface theory, and, given the relation between the double scaling variable ξ and the continuum cosmological constant, the continuum limit of the discretized model is indeed well-defined. This suggests that by restricting our "fermionic gravity" model to $\xi > 0$, we could define a model of two-dimensional quantum gravity with the canonical characteristics order by order in the genus expansion, but whose full asymptotic series makes perfect sense and provides a well-defined non-perturbative definition of the theory.

In the next and final section we will argue that this is indeed the case. In the usual Hermitian one-matrix models, the problem with the double scaling limit, in which the original matrix integral can only be defined by an analytical continuation, can be traced back to an instability in the corresponding double-scaled eigenvalue model. The non-compactness of the double-scaled eigenvalue space $\mathbb{R}$ and the form of the effective potential for the eigenvalues demonstrates that the definition of the critical points of the model is unstable to the tunneling of eigenvalues into a different configuration [27]. This leads to instanton solutions in the single-well eigenvalue model, and thereby explains the complexity of the free energy. However, in the present situation the Grassmann matrix integral is completely well-defined at finite N, in contrast to the Hermitian cases, and there is no reason *a priori* to expect that the model becomes unstable in any way at large N. Indeed, with the mapping of the fermionic matrix model onto a unitary matrix model, we see that the matrix integral is determined by an eigenvalue space whose topology is that of a circle. Being a compact space, there is no asymptotic behaviour in the effective eigenvalue action, and thus the compactness could have the effect of eliminating the eigenvalue tunneling problem. We see then that the unitary representation of the fermionic matrix model provides, at least naively, an eigenvalue description of the random surface model which reflects the reasons why the matrix integral is superior to its Hermitian counterparts. We shall tackle this problem using an operator theoretic approach which gives the fermionic analog of the (generalized) KdV hierarchical structure of the Hermitian one-matrix models [4, 28]. The integrable flows that we shall find are similar to those of the bosonic models, but with some very important changes. The most important difference will be the absence of a translational symmetry in eigenvalue space. This feature can be interpreted as allowing one to restrict the double-scaled specific heat to positive values of the cosmological constant ξ. It may then follow that the partition function of the fermionic matrix model

serves as an unambiguous definition of that for two-dimensional quantum gravity.

We stress that the resulting Borel summability of the fermionic case is very different from that of Borel summable bosonic models, such as the conformal field theory of the Yang-Lee edge singularity in which the lattice system couples to an imaginary magnetic field and is therefore non-unitary [4]. The coefficients of the genus expansion of the adjoint fermion one-matrix model are all real-valued. The stability of the fermionic model is also quite distinct from the stabilizations provided by supersymmetric and stochastic quantization methods [29], because these latter models violate the KdV flow structure of two-dimensional quantum gravity at a non-perturbative level [30]. In the present case, we will see that the appropriate integrable hierarchical structure is a deformation of that for gravity, and so the KdV flow structure continues to hold in a certain sense. The precise meaning of this deformation comes from the worldsheet interpretation of the fermionic one-matrix model. This is the model of fermionic gravity that we alluded to in section 2.1, wherein "Penner matter states" are located at the vertices of the corresponding surface discretizations.

5 Operator Formalism

The discussion at the end of the previous section motivates the development of an operator approach to analyse more carefully the properties of the adjoint fermion one-matrix model within the orthogonal polynomial formalism. The main purpose is two-fold. First of all, it will allow us to relate the partition function of the fermionic matrix model to the τ-function of an integrable hierarchy [28, 31, 15]. The usefulness of this association is that it partitions the solutions of the fermionic matrix model into universality classes which are related to well-known integrable hierarchies, and thereby formulates the partition function in an invariant way as the solution to a set of differential equations, in particular in the double scaling limit. Secondly, given this relationship, we will obtain a clear geometric picture of the origin of the alternating, Borel summable genus expansion of the model. This will allow us to clarify the description of the string susceptibility as a well-defined, non-perturbative generating function for the fermionic gravity model, as alluded to at the end of the previous section, and at the same time it will present the appropriate generalization of the KdV flow structure that characterizes the gravity model, lending a complementary characterization to the worldsheet description. In the following we will begin by introducing the formalism and deriving the main equations that will be required. We will then identify the integrable hierarchy of which the fermionic partition function is a τ-function, and derive the Virasoro constraints which must be satisfied by the flows.

5.1 String Equations

We begin by introducing multiplication operators $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ defined on the space $\mathbb{C}[z]\otimes\mathbb{C}[\bar{z}]$ by

$$(\mathbf{Q}\,\Phi_m)(z) \equiv z\,\Phi_m(z) \qquad , \qquad (\overline{\mathbf{Q}}\,\Lambda_m)(z) \equiv \frac{1}{z}\,\Lambda_m(z) \tag{5.1}$$

and differentiation operators $\mathbf{P}$ and $\overline{\mathbf{P}}$ by

$$(\mathbf{P}\,\Phi_m)(z) \equiv \Phi'_m(z) \qquad , \qquad (\overline{\mathbf{P}}\,\Lambda_m)(z) \equiv z^2\,\Lambda'_m(z) \tag{5.2}$$

The operators $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ are related to each other in a very simple way. From (5.1) it follows that

$$\Big\langle \mathbf{Q}\,\Phi_n(z) \,\Big|\, \overline{\mathbf{Q}}\,\Lambda_m(z)\Big\rangle = \Big\langle \Phi_n(z) \,\Big|\, \Lambda_m(z)\Big\rangle \tag{5.3}$$

and so $\mathbf{Q}$ is an invertible operator with inverse

$$\mathbf{Q}^{-1} = \overline{\mathbf{Q}}^{\dagger} \tag{5.4}$$

where the adjoint $\mathbf{O}^{\dagger}$ of any operator $\mathbf{O}$ on $\mathbb{C}[z]\otimes\mathbb{C}[\bar{z}]$ is defined by

$$\Big\langle F(z) \,\Big|\, (\mathbf{O}\,G)(\bar{z})\Big\rangle = \Big\langle (\mathbf{O}^{\dagger}\,F)(z) \,\Big|\, G(\bar{z})\Big\rangle \tag{5.5}$$

By writing $[z\,\Phi_m(z)]'$ in two different ways as $[(\mathbf{Q}\,\Phi_m)(z)]'$ and $\Phi_m(z) + z\,\Phi'_m(z)$, we can infer the canonical commutation relation

$$\Big[\mathbf{P}\,,\,\mathbf{Q}\Big] = \mathbf{1} \tag{5.6}$$

Similarly, by considering $z^2\,[\frac{1}{z}\,\Lambda_m(z)]'$ we arrive at $\Big[\overline{\mathbf{Q}}\,,\,\overline{\mathbf{P}}\Big] = \mathbf{1}$, which with the unitarity condition (5.4) implies the canonical commutator

$$\Big[\overline{\mathbf{P}}^{\dagger}\,,\,\mathbf{Q}^{-1}\Big] = \mathbf{1} \tag{5.7}$$

Geometrically, the operator $\mathbf{P}$ is the canonical conjugate of the operator generating clockwise unit shifts on the circle, while $\overline{\mathbf{P}}$ serves as the canonical conjugate of that which generates counterclockwise shifts. The operators introduced above are not all independent, but are related to each other through a Schwinger-Dyson equation. This follows from the inner product

$$\begin{aligned}\Big\langle (\mathbf{P}\,\Phi_n)(z) \,\Big|\, \Lambda_m(z)\Big\rangle &= \Big\langle \Phi'_n(z) \,\Big|\, \Lambda_m(z)\Big\rangle \\ &= (N+1)\Big\langle \Phi_n(z) \,\Big|\, \tfrac{1}{z}\,\Lambda_m(z)\Big\rangle - N\Big\langle V'(z)\,\Phi_n(z) \,\Big|\, \Lambda_m(z)\Big\rangle \\ &\quad - \Big\langle \Phi_n(z) \,\Big|\, \Lambda'_m(z)\Big\rangle \end{aligned} \tag{5.8}$$

which implies the operator identity

$$\mathbf{P} + \overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-2} = (N+1)\,\mathbf{Q}^{-1} - N\,V'(\mathbf{Q}) \tag{5.9}$$

The equations (5.6), (5.7) and (5.9) define the string equations of the adjoint fermion one-matrix model.

Let us now describe some basic properties of the operators introduced above. For this, we first rewrite their actions on the *same* basis Φ_n of the space of polynomials in a single variable on the circle. We have

$$\begin{aligned}(\mathbf{Q}^{\pm 1}\,\Phi_n)(z) &= \sum_{m\geq 0}\Big[\mathbf{Q}^{\pm 1}\Big]_{nm}\,\Phi_m(z)\\ (\mathbf{P}\,\Phi_n)(z) &= \sum_{m\geq 0}\Big[\mathbf{P}\Big]_{nm}\,\Phi_m(z)\\ (\overline{\mathbf{P}}^{\dagger}\,\Phi_n)(z) &= \sum_{m\geq 0}\Big[\overline{\mathbf{P}}\Big]_{nm}\,\Phi_m(z)\end{aligned} \tag{5.10}$$

We denote by $\mathbf{O}_+$ the upper triangular part, including the main diagonal, of the discrete operator $\mathbf{O}$ when represented in the basis of polynomials in (5.10). Then $\mathbf{O}_- = \mathbf{O} - \mathbf{O}_+$ is the lower triangular part of $\mathbf{O}$.

From the definition (5.1) it follows that the operator $\mathbf{Q}$ defines a Jacobi matrix, because

$$[\mathbf{Q}]_{nm} = 0 \qquad \text{for} \;\; m-n>1 \tag{5.11}$$

In particular, its pure upper triangular part is given by

$$\mathbf{Q}_+ - \sum_{n\geq 0}[\mathbf{Q}]_{nn}\,\mathbf{E}_{n,n} = \sum_{n\geq 0}\mathbf{E}_{n,n+1} \tag{5.12}$$

where $\mathbf{E}_{n,m}$ are the step operators with matrix elements $[\mathbf{E}_{n,m}]_{k\ell} = \delta_{kn}\,\delta_{\ell m}$. The remaining matrix elements of $\mathbf{Q}$ are given by $[\mathbf{Q}]_{nm} = \mathcal{P}_m^{(n)}$ in (3.26), along with the recurrence relations derived in section 3. Similarly, the operator $\mathbf{Q}^{-1}$ defines a Jacobi matrix in the basis (5.10) because

$$\Big[\mathbf{Q}^{-1}\Big]_{nm} = 0 \qquad \text{for} \;\; n-m>1 \tag{5.13}$$

Since

$$h_{n-1}\Big[\mathbf{Q}^{-1}\Big]_{n,n-1} = \Big\langle (\overline{\mathbf{Q}}^{\dagger}\,\Phi_n)(z)\;\Big|\;\Lambda_{n-1}(z)\Big\rangle = \Big\langle \Phi_n(z)\;\Big|\;\tfrac{1}{z}\,\Lambda_{n-1}(z)\Big\rangle = h_n \tag{5.14}$$

it follows that its lower triangular part is given by

$$\mathbf{Q}_-^{-1} = \sum_{n\geq 1} R_n\,\mathbf{E}_{n,n-1} \tag{5.15}$$

The remaining matrix elements of $\mathbf{Q}^{-1}$ are given by $[\mathbf{Q}^{-1}]_{nm} = \mathcal{L}_n^{(m)}$ in (3.31). Furthermore, from the definition (5.2) we see that $\mathbf{P} = \mathbf{P}_-$ is purely lower triangular with

$$[\mathbf{P}]_{n,n-1} = n \tag{5.16}$$

and since

$$h_{n+1}\Big[\overline{\mathbf{P}}\Big]_{n,n+1} = \Big\langle (\overline{\mathbf{P}}^{\dagger}\,\Phi_n)(z)\;\Big|\;\Lambda_{n+1}(z)\Big\rangle = \Big\langle \Phi_n(z)\;\Big|\;z^2\,\Lambda'_{n+1}(z)\Big\rangle = -(n+1)h_n \tag{5.17}$$

it follows that $\overline{\mathbf{P}}^\dagger = \overline{\mathbf{P}}^\dagger_+$ is purely upper triangular, $\left[\overline{\mathbf{P}}\right]_{nn} = 0$, with

$$\left[\overline{\mathbf{P}}\right]_{n,n+1} = -\frac{n+1}{R_{n+1}} \tag{5.18}$$

Note that, in this basis, the $(n, n-1)$ matrix element of the Schwinger-Dyson equation (5.9) coincides with (3.36). This follows from the structure of the matrix elements

$$h_{n-1}\left[\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-2}\right]_{n,n-1} = \left\langle \Phi_n(z) \,\middle|\, \Lambda'_{n-1}(z)\right\rangle = -(n-1)h_n \tag{5.19}$$

which implies that $\left[\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-2}\right]_{n,n-1} = -(n-1)R_n$.

5.2 Flow Equations

We will now derive a system of flow equations for the operators above and the partition function (3.7), which will enable us to identify the particular integrable hierarchy at work here. We shall describe the evolutions with respect to the discrete set of "time" variables $t_k \equiv N g_k/k$ of the generic potential (1.2). Consider the relation

$$\frac{\partial}{\partial t_k}\left\langle \Phi_n(z) \,\middle|\, \Lambda_{n-m}(z)\right\rangle \;=\; 0 \;=\; \left\langle z^k\,\Phi_n(z) \,\middle|\, \Lambda_{n-m}(z)\right\rangle + \left\langle \tfrac{\partial}{\partial t_k}\Phi_n(z) \,\middle|\, \Lambda_{n-m}(z)\right\rangle \tag{5.20}$$

where $1 \le m \le n$ and we have used the fact that $\frac{\partial \Lambda_{n-m}(z)}{\partial t_k}$ is a polynomial of degree at most $n-m-1$. This relation leads immediately to the evolution equation

$$\frac{\partial \Phi_n(z)}{\partial t_k} = -\sum_{m=0}^{n-1}\left[\mathbf{Q}^k\right]_{nm}\Phi_m(z) = -\left(\mathbf{Q}^k_-\,\Phi_n\right)(z) \tag{5.21}$$

By taking time derivatives of the equations (5.10), we then arrive at the discrete operator flow equations

$$\frac{\partial \mathbf{Q}}{\partial t_k} \;=\; \left[\mathbf{Q}\,,\,\mathbf{Q}^k_-\right] \tag{5.22}$$

$$\frac{\partial \mathbf{P}}{\partial t_k} \;=\; \left[\mathbf{P}\,,\,\mathbf{Q}^k_-\right] \tag{5.23}$$

$$\frac{\partial \mathbf{Q}^{-1}}{\partial t_k} \;=\; \left[\mathbf{Q}^{-1}\,,\,\mathbf{Q}^k_-\right] \tag{5.24}$$

$$\frac{\partial \overline{\mathbf{P}}^\dagger}{\partial t_k} \;=\; \left[\overline{\mathbf{P}}^\dagger\,,\,\mathbf{Q}^k_-\right] \tag{5.25}$$

The flow equations (5.22) constitute a discrete KP hierarchy [31]. It is straightforward to show that these flows are mutually commutative, so that this hierarchy is in fact integrable. This follows from taking time derivatives of (5.10) and using (5.21) to get

$$\frac{\partial\left[\mathbf{Q}^k\right]_{mn}}{\partial t_r} = \sum_{\ell=m-k}^{n-1}\left[\mathbf{Q}^k\right]_{m\ell}\left[\mathbf{Q}^r\right]_{\ell n} - \sum_{\ell=m+1}^{n+k}\left[\mathbf{Q}^r\right]_{m\ell}\left[\mathbf{Q}^k\right]_{\ell n} \tag{5.26}$$

which leads to the Zakharov-Shabat equations

$$\frac{\partial \mathbf{Q}_-^m}{\partial t_n} - \frac{\partial \mathbf{Q}_-^n}{\partial t_m} + \left[\mathbf{Q}_-^m\,,\,\mathbf{Q}_-^n\right] = 0 \tag{5.27}$$

The matrix model actually defines a reduced, generalized KP hierarchy, because the flow equations are to be supplemented with the constraints imposed by the string equations.

The knowledge of the solutions $\mathbf{Q}$ to (5.22) completely determines the partition function of the fermionic matrix model. To see this, we consider the $(n, n-1)$ matrix element of the flow equation (5.24), which using (5.13) and (5.15) leads to the differential equation

$$\frac{\partial \log R_n}{\partial t_k} = \left[\mathbf{Q}^k\right]_{nn} - \left[\mathbf{Q}^k\right]_{n-1,n-1} \tag{5.28}$$

We can use (3.7) and (5.28) to obtain the flow equation for the partition function

$$\frac{\partial \log Z_N}{\partial t_k} = \mathrm{Tr}_{(N)}\left(\mathbf{Q}^k\right) \tag{5.29}$$

where we have defined the N dimensional trace

$$\mathrm{Tr}_{(N)}(\mathbf{O}) = \sum_{n=0}^{N-1} [\mathbf{O}]_{nn} \tag{5.30}$$

Using the KP equation (5.22) and the Jacobi property (5.11) of the $\mathbf{Q}$ operator, we have

$$\frac{\partial [\mathbf{Q}]_{nn}}{\partial t_k} = \left[\mathbf{Q}^k\right]_{n+1,n} - \left[\mathbf{Q}^k\right]_{n,n-1} \tag{5.31}$$

which, on using (5.28) with $k = 1$, leads to

$$\frac{\partial^2 \log Z_N}{\partial t_1 \partial t_k} = \left[\mathbf{Q}^k\right]_{N,N-1} \tag{5.32}$$

This procedure can be iterated to determine any number of time derivatives of $\log Z_N$ in terms of the $\mathbf{Q}$ operators. Knowing $\mathbf{Q}$ therefore determines the free energy of the matrix model up to an overall integration constant.

5.3 Fermionic τ-Function

It is clear from (5.32) that the partition function Z_N is a τ-function for the discrete KP hierarchy (5.22) [31]. Let us introduce the Baker-Akhiezer functions

$$\Psi_n[\,\vec{t}\,; z] = \frac{\mathrm{e}^{\frac{N}{2}V(z)}}{z^{\frac{N+1}{2}}}\,\frac{\Phi_n(z)}{\sqrt{h_n}} \tag{5.33}$$

and their conjugates

$$\overline{\Psi}_n[\,\vec{t}\,; z] = \frac{\mathrm{e}^{\frac{N}{2}V(z)}}{z^{\frac{N+1}{2}}}\,\frac{\Lambda_n(z)}{\sqrt{h_n}} \tag{5.34}$$

which are bi-orthonormal in the standard line measure of the complex plane,

$$\oint_{z=0} dz\ \Psi_n[\,\vec{t}\,;z]\,\overline{\Psi}_m[\,\vec{t}\,;z] = \delta_{nm} \tag{5.35}$$

The operator $\mathbf{Q}$ is a discrete Lax type operator which has eigenfunctions $\Psi_n[\,\vec{t}\,;z]$ with eigenvalue z,

$$\mathbf{Q}\,\Psi_n[\,\vec{t}\,;z] = z\,\Psi_n[\,\vec{t}\,;z] \tag{5.36}$$

The function $Z_N[\,\vec{t}\,]$ is then the generating function for the Lax operator $\mathbf{Q}$, in the sense that $\mathbf{Q}$ can be reconstructed from a knowledge of Z_N. In fact, if we introduce the discrete τ-function $\tau_n[\,\vec{t}\,]$ for this hierarchy by the formula (4.1) [15], then from (3.7) we see that the partition function of the fermionic matrix model is given by

$$Z_N[\,\vec{t}\,] = \tau_N[\,\vec{t}\,] \tag{5.37}$$

Formally, the τ-function is a section of the determinant line bundle over a Sato Grassmannian associated with Riemann surfaces of the spectral parameters z [31]. It may be characterized as the unique function depending on the times t_k and satisfying the given hierarchy of constrained differential equations. It is the generating function for solutions of the generalized KdV equations. Thus once the τ-function is known, everything about the matrix model is likewise known. The usefulness of the relationship (5.37) is that there are a large number of identities satisfied by $\tau_N[\,\vec{t}\,]$ that can be used to characterize the partition function of the fermionic matrix model.

The flow equations (5.22) or (5.26) constitute the Lax representation of a discrete integrable hierarchy of differential equations of Toda type [32]. It is tempting therefore to characterize the fermionic partition function as the τ-function of a certain reduction of the integrable Toda chain. However, as we will now demonstrate, this is not the case. By equating the constant terms on both sides of the flow equation (5.21) for $k = 1$ and using (3.26) and (3.27), we have

$$\frac{\partial p_{n,0}}{\partial t_1} = \frac{1 - R_{n+1}}{l_{n+1,0}} \sum_{m=0}^{n-1} p_{m,0}\, l_{m,0} \prod_{k=m+1}^{n} R_k \tag{5.38}$$

By iterating the relation (3.30) using (3.27) it follows that

$$p_{n,0}\, l_{n,0} = (-1)^{n+1}\,(R_n - 1) \tag{5.39}$$

where we have defined $R_0 \equiv 0$. Substituting (5.39) into (5.38) we obtain

$$\frac{\partial p_{n,0}}{\partial t_1} = p_{n+1,0}\, R_n = p_{n+1,0}\left(1 + (-1)^{n+1}\, p_{n,0}\, l_{n,0}\right) \tag{5.40}$$

Similarly, one can derive the evolution equation

$$\frac{\partial l_{n,0}}{\partial t_1} = -l_{n-1,0}\left(1 + (-1)^{n+1}\, p_{n,0}\, l_{n,0}\right) \tag{5.41}$$

The flow equations (5.40) and (5.41) are the first members of an integrable Hamiltonian system known as the Toeplitz chain hierarchy [24]. The general evolution equations may be similarly derived using the flow equations of the previous subsection and are given by the Hamilton equations of motion

$$\frac{\partial p_{n,0}}{\partial t_k} = \Big(1+(-1)^{n+1}\, p_{n,0}\, l_{n,0}\Big)\, \frac{\partial H_k}{\partial l_{n,0}} \qquad , \qquad \frac{\partial l_{n,0}}{\partial t_k} = -\Big(1+(-1)^{n+1}\, p_{n,0}\, l_{n,0}\Big)\, \frac{\partial H_k}{\partial p_{n,0}} \tag{5.42}$$

for the system of Hamiltonians

$$H_k = -\frac{1}{k}\, \mathrm{Tr}_{(N)}\left(\mathbf{Q}^k\right) = -\frac{1}{k}\, \frac{\partial \log Z_N}{\partial t_k} \qquad , \qquad k \geq 1 \tag{5.43}$$

and the symplectic structure

$$\omega = \sum_{n\geq 1} \frac{dp_{n,0} \wedge dl_{n,0}}{1+(-1)^{n+1}\, p_{n,0}\, l_{n,0}} \tag{5.44}$$

The Toeplitz chain is a particular reduction of the *two-dimensional* Toda lattice hierarchy, the latter of which characterizes generic matrix integrals [28, 15]. Thus the integrable hierarchy to which the adjoint fermion one-matrix model belongs is not the reduction of a one-dimensional (Toda) hierarchy, but rather of a two-dimensional one. This is of course anticipated from the lattice interpretation of the fermionic matrix model and its associated doubling of degrees of freedom due to the Penner matter states.

However, the Toeplitz chain hierarchy can be viewed as a deformation of the standard Toda chain hierarchy. To see this, we use (5.28) for $k = 1$ and (3.26) for $m = n$ to obtain the flow equation

$$\frac{\partial \log h_n}{\partial t_1} = [\mathbf{Q}]_{nn} = Q_n\Big(R_{n+1} - 1\Big) \tag{5.45}$$

On the other hand, by using (5.31) for $k = 1$ and (3.26) for $m = n-1$ we have

$$\frac{\partial [\mathbf{Q}]_{nn}}{\partial t_1} = Q_n\Big[Q_{n+1}R_{n+1}(R_{n+2}-1) - Q_{n-1}R_n(R_{n+1}-1)\Big] \tag{5.46}$$

The two flow equations (5.45) and (5.46) may be combined together to give

$$\frac{\partial^2 \log h_n}{\partial t_1^2} = \frac{\partial \log h_n}{\partial t_1}\left(\frac{R_{n+1}}{R_{n+1}-1}\, \frac{\partial \log h_{n+1}}{\partial t_1} - \frac{R_n}{R_n - 1}\, \frac{\partial \log h_{n-1}}{\partial t_1}\right) \tag{5.47}$$

By introducing time-dependent functions $q_n[\,\vec{t}\,]$ through

$$h_n[\,\vec{t}\,] = (-1)^n\, \varepsilon^{2n}\ \mathrm{e}^{t_1/\varepsilon}\ \mathrm{e}^{q_n[\,\vec{t}\,]} \tag{5.48}$$

where ε is a time-independent parameter, we can write (5.47) as

$$\begin{aligned} \frac{\partial^2 q_n}{\partial t_1^2} \;\; = \;\; & \left(1+\varepsilon\, \frac{\partial q_n}{\partial t_1}\right)\left(1+\varepsilon\, \frac{\partial q_{n+1}}{\partial t_1}\right)\, \frac{\mathrm{e}^{q_{n+1}-q_n}}{1+\varepsilon^2\ \mathrm{e}^{q_{n+1}-q_n}} \\ & - \left(1+\varepsilon\, \frac{\partial q_{n-1}}{\partial t_1}\right)\left(1+\varepsilon\, \frac{\partial q_n}{\partial t_1}\right)\, \frac{\mathrm{e}^{q_n-q_{n-1}}}{1+\varepsilon^2\ \mathrm{e}^{q_n-q_{n-1}}} \end{aligned} \tag{5.49}$$

The differential equation (5.49) is the first member of an integrable hierarchy known as the *relativistic* Toda chain [33]. In the "non-relativistic" limit $\varepsilon \to 0$, it reduces to the Toda chain hierarchy which characterizes generic Hermitian one-matrix models. Thus, the partition function of the adjoint fermion one-matrix model is a τ-function of not the Toda chain hierarchy, but rather of its deformation to the relativistic Toda chain. The latter chain is itself a reduction of the two-dimensional Toda lattice. From this point of view, the fermionic matrix model may be thought of as a "deformation" of the Hermitian one-matrix model, the role of the deformation being played by the Penner interaction.

The remaining differential equations of the relativistic Toda chain hierarchy are given by the Lax equations (5.26), by (5.28), and by exploiting the Jacobi properties (5.11) and (5.12) of the operator $\mathbf{Q}$. It is important to realize that, because the present model contains only a single set $\vec{t}$ of times, the ordinary Toda lattice does not itself appear in the hierarchy satisfied by the partition function. Thus the standard Toda lattice structure which underlies all integrable models (and in particular matrix integrals) only appears in a very subtle way through the reductions obtained above by eliminating one set of its time variables. This reduced structure leads to a certain degeneracy in the τ-function as compared to the usual two-dimensional Toda lattice structure. Of course, the relations to integrable hierarchies described in this subsection are only 'kinematical', as they merely rely on the very basic recursion properties satisfied by the orthogonal polynomials. To incorporate the 'dynamical' aspects, and in particular the effects of the Penner potential, we need to impose the constraints implied by the string equations. Indeed, there are many τ-function solutions of the above hierarchies, and the string equations select the one appropriate to describe the nonperturbative dynamics of the adjoint fermion one-matrix model. This is the topic of the next subsection.

5.4 Virasoro Constraints

We will now begin examining the constraints imposed on the fermionic τ-function $Z_N[\,\vec{t}\,]$ as dictated by the string equations. Proceeding as in (5.8) for the operators $\mathbf{P}\,\mathbf{Q}^{n+1}$, $n \geq -1$, we can express the string equation (5.9) as an infinite system of equations

$$\mathrm{Tr}_{(N)}\left[\left(\mathbf{P} + \overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-2} - (N+1)\,\mathbf{Q}^{-1} + \sum_{k\geq 1} k t_k\,\mathbf{Q}^{k-1}\right)\mathbf{Q}^{n+1}\right] = 0 \qquad , \qquad n \geq -1 \quad (5.50)$$

The strategy now is to rewrite the equations of motion (5.50) using the flow equations (5.29) and (5.32). In this way, the string equations will be represented as the annihilation of the partition function by a system of differential operators in the coupling constants of the potential V. In this subsection we will deal with the cases $n \geq 0$.

Let us start with the $n = 0$ string equation of (5.50). Using the canonical commutation relations (5.6) and the matrix elements

$$h_n[\mathbf{Q}\,\mathbf{P}]_{nn} = \Big\langle z\,\Phi_n'(z) \;\Big|\; \Lambda_n(z)\Big\rangle = n h_n \quad (5.51)$$

we may compute the first trace in (5.50) for $n = 0$ as

$$\mathrm{Tr}_{(N)}(\mathbf{P}\,\mathbf{Q}) = N + \sum_{n=0}^{N-1} n = \frac{N(N+1)}{2} \tag{5.52}$$

The second trace may be similarly computed by using

$$h_n \left[\overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-1}\right]_{nn} = \left\langle \Phi_n(z) \,\middle|\, z\,\Lambda_n'(z) \right\rangle = -n h_n \tag{5.53}$$

to get

$$\mathrm{Tr}_{(N)}\left(\overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-1}\right) = -\frac{N(N-1)}{2} \tag{5.54}$$

By substituting (5.52) and (5.54), and using (5.29), we can represent the $n = 0$ constraint of the system (5.50) as the flow equation

$$\left(\sum_{k\geq 1} k t_k \frac{\partial}{\partial t_k} - N^2\right) Z_N[\,\vec{t}\,] = 0 \tag{5.55}$$

Next, let us consider the $n = 1$ constraint of (5.50). For the first trace, we use (3.5) and (3.6) to compute the matrix elements

$$\begin{aligned} h_n \left[\mathbf{Q}^2\,\mathbf{P}\right]_{nn} &= \left\langle z^2\,\Phi_n'(z) \,\middle|\, \Lambda_n(z) \right\rangle \\ &= \left\langle n\,z^{n+1} - (n-1)p_{n,n-1}\,z^n + O\left(z^{n-1}\right) \,\middle|\, \Lambda_n(z) \right\rangle \\ &= \left[n p_{n+1,n} - (n-1)p_{n,n-1}\right] h_n \end{aligned} \tag{5.56}$$

Using (3.16) and the canonical commutator (5.6) we thereby arrive after a little algebra at

$$\mathrm{Tr}_{(N)}\left(\mathbf{P}\,\mathbf{Q}^2\right) = 2\,\mathrm{Tr}_{(N)}(\mathbf{Q}) + \sum_{n=1}^{N-1}\left(n[\mathbf{Q}]_{nn} + \mathrm{Tr}_{(n)}(\mathbf{Q})\right) = (N+1)\,\mathrm{Tr}_{(N)}(\mathbf{Q}) \tag{5.57}$$

Since $\left[\overline{\mathbf{P}}^{\dagger}\right]_{nn} = 0$, upon substituting (5.57) into (5.50) we find that the terms linear in $\mathbf{Q}$ cancel out and applying the flow equation (5.29) we see that the $n = 1$ constraint can be written as

$$\sum_{k\geq 1} k t_k \frac{\partial}{\partial t_{k+1}}\; Z_N[\,\vec{t}\,] = 0 \tag{5.58}$$

Now we move on to the $n = 2$ string equation of (5.50). Again using (3.5) and (3.6) we may compute

$$\begin{aligned} h_n \left[\mathbf{Q}^3\,\mathbf{P}\right]_{nn} &= \left\langle z^3\,\Phi_n'(z) \,\middle|\, \Lambda_n(z) \right\rangle \\ &= \left\langle n\,z^{n+2} - (n-1)p_{n,n-1}\,z^{n+1} - (n-2)p_{n,n-2}\,z^n + O\left(z^{n-1}\right) \,\middle|\, \Lambda_n(z) \right\rangle \\ &= \left[n(p_{n+2,n} + p_{n+2,n+1}p_{n+1,n}) - (n-2)p_{n,n-2} - (n-1)p_{n+1,n}p_{n,n-1}\right] h_n \end{aligned} \tag{5.59}$$

Using (3.16) we can iterate the relation (3.14) for $k = n - 2$ to get

$$p_{n,n-2} = \sum_{k=1}^{n-1} \Big([\mathbf{Q}]_{k,k-1} + [\mathbf{Q}]_{kk} \, \mathrm{Tr}_{(k)}(\mathbf{Q}) \Big) \tag{5.60}$$

Substituting (5.60) and (3.16) into (5.59) and using (5.6), we arrive after some algebra at

$$\begin{aligned} \mathrm{Tr}_{(N)} \left(\mathbf{P} \, \mathbf{Q}^3 \right) &= 3 \, \mathrm{Tr}_{(N)} \left(\mathbf{Q}^2 \right) + (N-1) [\mathbf{Q}]_{N,N-1} + (N+1) \Big(\mathrm{Tr}_{(N)}(\mathbf{Q}) \Big)^2 \\ &\quad + (2N-3) \sum_{n=1}^{N-1} \Big([\mathbf{Q}]_{n,n-1} - [\mathbf{Q}]_{nn} \, \mathrm{Tr}_{(n)}(\mathbf{Q}) \Big) \end{aligned} \tag{5.61}$$

The sums in (5.61) can be simplified by using the Jacobi property (5.11) of the operator $\mathbf{Q}$ to get

$$\mathrm{Tr}_{(N)} \Big(\mathbf{Q}^2 \Big) = [\mathbf{Q}]_{N,N-1} + 2 \sum_{n=1}^{N-1} [\mathbf{Q}]_{n,n-1} + \sum_{n=0}^{N-1} \Big([\mathbf{Q}]_{nn} \Big)^2 \tag{5.62}$$

and in this way we arrive at an expression for the first trace in the $n = 2$ equation of (5.50),

$$\mathrm{Tr}_{(N)} \left(\mathbf{P} \, \mathbf{Q}^3 \right) = \left(N + \tfrac{3}{2} \right) \, \mathrm{Tr}_{(N)} \left(\mathbf{Q}^2 \right) + \tfrac{1}{2} \, [\mathbf{Q}]_{N,N-1} + \tfrac{1}{2} \left(\mathrm{Tr}_{(N)}(\mathbf{Q}) \right)^2 \tag{5.63}$$

For the second trace in (5.50), we use (3.5) to compute the matrix elements

$$\begin{aligned} h_n \left[\overline{\mathbf{P}}^\dagger \, \mathbf{Q} \right]_{nn} &= \Big\langle \Phi_n(z) \, \Big| \, z^3 \, \Lambda_n'(z) \Big\rangle \\ &= \Big\langle \Phi_n(z) \, \Big| \, -n \, z^{-(n-2)} + O\left(z^{-(n-3)} \right) + l_{n,1} \, z \Big\rangle \\ &= l_{n,1} \Big\langle z \, \Phi_n(z) \, \Big| \, 1 \Big\rangle \end{aligned} \tag{5.64}$$

Using (3.25) we obtain

$$\left[\overline{\mathbf{P}}^\dagger \, \mathbf{Q} \right]_{nn} = S_n Q_n (1 - R_{n+1}) \tag{5.65}$$

where we have defined

$$S_n = \frac{l_{n,1}}{l_{n,0}} \tag{5.66}$$

By equating the coefficients of the z^{-1} terms on both sides of the three-term recursion relation (3.32) for the Λ polynomials, we can write down an iterative equation for the coefficients (5.66),

$$S_n = R_n Q_{n-1} + S_{n+1} - Q_n \tag{5.67}$$

which has solution

$$S_n = Q_{n-1} R_n + \sum_{k=0}^{n-1} Q_k (1 - R_{k+1}) \tag{5.68}$$

Substituting (5.68) into (5.65) and using (3.26) for $m = n - 1$ and $m = n$, we therefore find

$$\left[\overline{\mathbf{P}}^\dagger \, \mathbf{Q} \right]_{nn} = [\mathbf{Q}]_{nn} \, \mathrm{Tr}_{(n)}(\mathbf{Q}) - [\mathbf{Q}]_{n,n-1} \tag{5.69}$$

Using the identity (5.62) we then arrive at an expression for the second trace in the $n = 2$ equation of (5.50),

$$\mathrm{Tr}_{(N)}\left(\overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}\right) = \tfrac{1}{2}\,[\mathbf{Q}]_{N,N-1} + \tfrac{1}{2}\left(\mathrm{Tr}_{(N)}(\mathbf{Q})\right)^2 - \tfrac{1}{2}\,\mathrm{Tr}_{(N)}\left(\mathbf{Q}^2\right) \tag{5.70}$$

Upon substitution of (5.63) and (5.70) into (5.50) for $n = 2$, we see that the terms quadratic in $\mathbf{Q}$ cancel out. The remaining terms can be simplified using the flow equations (5.29) and (5.32), which leads to the $n = 2$ constraint equation

$$\left(\sum_{k\geq 1} k t_k\,\frac{\partial}{\partial t_{k+2}} + \frac{\partial^2}{\partial t_1^2}\right) Z_N[\,\vec{t}\,] = 0 \tag{5.71}$$

This procedure can be easily generalized to higher orders $n \geq 3$. However, it is possible to determine the general relation from the first three equations (5.55), (5.58) and (5.71) by using the fact that the commutator bracket of the pertinent differential operators must also annihilate the partition function and thereby using commutators of the above operators to generate the higher order ones. The final result can be expressed in a more familiar form by introducing the zeroth time t_0 through the flow equation

$$\frac{\partial Z_N}{\partial t_0} = N Z_N \tag{5.72}$$

for the partition function.

In this way we find that the string equations (5.50) for $n \geq 0$ can be written as the set of discrete Virasoro constraints

$$L_n\,Z_N[\,\vec{t}\,] = 0 \qquad , \qquad n \geq 0 \tag{5.73}$$

where the second order linear differential operators

$$L_n = \sum_{k\geq 1} k t_k\,\frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial}{\partial t_k}\,\frac{\partial}{\partial t_{n-k}} - 2N\,\frac{\partial}{\partial t_n} \tag{5.74}$$

satisfy the Witt algebra

$$\Big[L_n\,,\,L_m\Big] = (n-m)\,L_{n+m} \tag{5.75}$$

for $n, m \in \mathbb{Z}^+$. These Virasoro constraints are the usual ones for the fermionic matrix model [8, 9] and they are identical to those of the Hermitian Penner matrix model [34] as defined in (1.3) with $\alpha = -2$. They represent the full system of equations of motion of the matrix model and can be equivalently derived by demanding the invariance of the matrix integral under arbitrary changes of the matrix variables [8, 9]. The first set of operators in (5.74) comes from the variation of the potential V in the action, while the second one comes from the Jacobian of the change of matrix integration measure. The first two terms in (5.74) therefore coincide with the standard Virasoro generators of generic Hermitian one-matrix models [28]. The last operator in (5.74) comes from the variation of the logarithmic Penner interaction.

It is a remarkable fact that the loop equations of the adjoint fermion one-matrix model (1.1), the Hermitian Penner one-matrix model ((1.3) with $\alpha = -2$), and the unitary Penner matrix model (1.4) are all equivalent. However, the equivalence between the Hermitian model and the other two matrix models only holds order by order in the $\frac{1}{N}$ expansions of the matrix integrals. Although the models are equivalent at $N = \infty$, beyond leading order in the large N expansion the loop equations should be solved with different boundary conditions and the solutions are different. The fermionic and unitary equations should be solved with boundary conditions appropriate to a perturbative Gaussian limit, in order that the models admit interpretations in terms of random surfaces, while the Hermitian ones should be solved with boundary conditions appropriate to recover the Penner model of the discretized moduli space of Riemann surfaces [10, 16]. The distinction of the fermionic and unitary matrix integrals from the Hermitian one is evident from the structure of the finite N correlators (see appendix A). In fact, it is only in the large N limit that a full Virasoro symmetry is realized in these models, as well as an infinite hierarchy of generalized KP differential equations. This makes the large N limit a very natural ingredient of the fermionic matrix model, as is also apparent from its fat-graph interpretation of section 2.2.

5.5 Origin of the Painlevé Expansion

In the previous subsection we have shown that the Schwinger-Dyson equations of the fermionic matrix model are generated by the positive Borel subalgebra of a Virasoro algebra of vanishing central charge. There is a nice algebraic way to characterize the effect of the Penner interaction potential in the Virasoro generators (5.74). Let us write them as $L_n = L_n^{\mathrm{H}} + T_n$, where L_n^{H} are the standard Virasoro generators of a Hermitian one-matrix model with potential V, and

$$T_n = -2N \frac{\partial}{\partial t_n} \tag{5.76}$$

are the generators of translations $t_n \mapsto t_n - 2N\varepsilon$ in coupling constant space. Together these operators generate the symmetry algebra

$$\begin{aligned} \Big[T_n\,,\,T_m\Big] &= 0 \\ \Big[L_n^{\mathrm{H}}\,,\,L_m^{\mathrm{H}}\Big] &= (n-m)\,L_{n+m}^{\mathrm{H}} \\ \Big[T_n\,,\,L_m^{\mathrm{H}}\Big] &= -2Nn\,T_{n+m} \end{aligned} \tag{5.77}$$

The commutation relations (5.77) characterize a master symmetry of the generalized KdV hierarchy [31]. We see then that the Virasoro constraints of the fermionic matrix model constitute a deformation of those for Hermitian one-matrix models by the master symmetry algebra generators of the KdV flow equations. Note that the particular symmetry generated by the operators (5.76) is restricted to translations in units of a discrete "lattice spacing" $-2N$, indicating what sort of reduction of this master symmetry is taken.

These Virasoro constraints describe the desired reduction of the fermionic τ-function, and they lead immediately to the desired geometrical interpretation of the Painlevé expansion of the adjoint fermion one-matrix model. The Painlevé equations may be characterized as following from a reduction of the τ-function of the integrable Toeplitz (or relativistic Toda) chain hierarchy, subject to the Virasoro symmetry obtained in the previous subsection. They yield a stability condition $L_n\,\tau_N = 0$ on the points of the model Grassmannian, which are associated with the Baker-Akhiezer functions (5.33), that select a particular class of transcendental solutions to the KP equations.

It is a standard fact that a set of operators of the form (5.74) can be embedded into a Virasoro algebra of non-vanishing central charge. However, an important aspect of the present approach to the fermionic matrix model is that there are in fact extra dynamical constraints imposed on the partition function which can be attributed to the invertibility of the Lax operator $\mathbf{Q}$. For instance, the constraint (5.50) makes perfect sense for $n = -1$, and the calculation of the trace of the operator $\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-2}$ proceeds in an analogous manner to that of (5.56) with the result

$$\mathrm{Tr}_{(N)}\left(\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-2}\right) = -(N-1)\,\mathrm{Tr}_{(N)}\left(\mathbf{Q}^{-1}\right) \tag{5.78}$$

Since $[\mathbf{P}]_{nn} = 0$, we see that on substitution of (5.78) into (5.50) for $n = -1$ we produce an extra non-vanishing term $-2N\,\mathrm{Tr}_{(N)}(\mathbf{Q}^{-1})$. This trace can be written in an operator form as follows. From (3.31) with $m = n$ and (5.45) we have

$$\left[\mathbf{Q}^{-1}\right]_{nn} = \Big(R_n - 1\Big)\Big(R_{n+1} - 1\Big)\left(\frac{\partial \log h_n}{\partial t_1}\right)^{-1} \tag{5.79}$$

This quantity is independent of the first time t_1, since on using the flow equation (5.24) for $k = 1$ and the Jacobi properties of the operators $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ we find $\partial[\mathbf{Q}^{-1}]_{nn}/\partial t_1 = 0$. This suggests defining a negative time t_{-1} such that $\mathrm{Tr}_{(N)}(\mathbf{Q}^{-1})$ is represented as an operator acting on the partition function as

$$\begin{aligned}
\frac{DZ_N}{Dt_{-1}} &= \sum_{n=1}^{N-1}\frac{1}{\tau_n\tau_{n+1}}\,\frac{\Big(\tau_{n+1}\tau_{n-1} - \tau_n^2\Big)\Big(\tau_{n+2}\tau_n - \tau_{n+1}^2\Big)}{\tau_n\,\frac{\partial\tau_{n+1}}{\partial t_1} - \tau_{n+1}\,\frac{\partial\tau_n}{\partial t_1}} \\
\frac{\partial}{\partial t_1}\,\frac{DZ_N}{Dt_{-1}} &= 0
\end{aligned} \tag{5.80}$$

where we have used (4.1). Then the constraint (5.50) for $n = -1$ can be written as

$$\left(\sum_{k\geq 2} k t_k\,\frac{\partial}{\partial t_{k-1}} + N t_1 - 2N\,\frac{D}{Dt_{-1}}\right) Z_N[\,\vec{t}\,] = 0 \tag{5.81}$$

In fact, there are infinitely many negatively moded constraints such as this, associated with higher powers $\mathbf{Q}^{-m}$, $m \geq 1$. However, the operators such as that in (5.81) which are thereby generated do not form a closed algebra among themselves or with the operators

(5.74). Indeed, this set of constraints arises from the previously mentioned reduction from a two-dimensional lattice (containing both negative and positive time parameters) to a (deformed) chain by eliminating all negative couplings. As usual, such reductions break the full Virasoro symmetry of the original integrable model and only the positive Borel subalgebra survives here as a symmetry of the matrix model. The negatively moded constraints are therefore merely an artifact of the lattice reduction or equivalently the chain deformation. In practical terms, all observables in the fermionic model are described by observables conjugate to positive times in the unitary matrix model. Observables conjugate to negative times exist but do not appear to play a role in the equivalence.

We can understand all of these matters more clearly by rewriting the Virasoro constraints (5.73) in terms of the Baker-Akhiezer functions (5.33). For this, we note first of all that (5.21) and (5.28) imply that they obey the flow equations

$$\frac{\partial \Psi_n[\vec{t};z]}{\partial t_k} = \frac{1}{2}\left(\mathbf{Q}_+^k - \mathbf{Q}_-^k - \left[\mathbf{Q}^k\right]_{nn}\right)\,\Psi_n[\vec{t};z] \tag{5.82}$$

Now let us compute the action of the scale transformation generator $z\,\frac{\partial}{\partial z}$ in eigenvalue space on the Baker-Akhiezer functions. We have

$$z\,\frac{\partial \Psi_n[\vec{t};z]}{\partial z} = \left(-\frac{N+3}{2} + \frac{N}{2}\,z\,V'(z)\right)\Psi_n[\vec{t};z] + \frac{\mathrm{e}^{\frac{N}{2}V(z)}}{z^{\frac{N+1}{2}}}\,\mathbf{P}\,\mathbf{Q}\,\Phi_n(z) \tag{5.83}$$

To evaluate the last term in (5.83), we have to take into account the triangularity of the operator $\mathbf{P}\,\mathbf{Q}$. From $\mathbf{P} = \mathbf{P}_-$ and the Jacobi property (5.11), we have

$$\mathbf{P}\,\mathbf{Q} = (\mathbf{P}\,\mathbf{Q})_- + \sum_{n\geq 0}[\mathbf{P}\,\mathbf{Q}]_{nn}\,\mathbf{E}_{n,n} \tag{5.84}$$

We can therefore evaluate the action of $\mathbf{P}\,\mathbf{Q}$ on $\Phi_n(z)$ by using the string equation (5.9) multiplied on the right by $\mathbf{Q}$, but eliminating the pure upper triangular part of both sides of the equation. Using (5.53) and the fact that

$$\left(\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-1}\right)_- = 0 \tag{5.85}$$

which can be derived similarly to the other matrix elements of this section, we can write (5.83) as

$$\begin{aligned} z\,\frac{\partial \Psi_n[\vec{t};z]}{\partial z} &= \frac{1}{2}\Big(2n+N-1\Big)\Psi_n[\vec{t};z] \\ &\quad + \frac{N}{2}\left[\Big(\mathbf{Q}\,V'(\mathbf{Q})\Big)_+ - \Big(\mathbf{Q}\,V'(\mathbf{Q})\Big)_- - \Big[\mathbf{Q}\,V'(\mathbf{Q})\Big]_{nn}\right]\Psi_n[\vec{t};z] \end{aligned} \tag{5.86}$$

The last term in (5.86) can be rewritten using the flow equation (5.82), leading to the eigenvalue equation

$$L_0[\vec{t};z]\,\Psi_n[\vec{t};z] = -n\,\Psi_n[\vec{t};z] \tag{5.87}$$

where we have defined the Virasoro operator

$$L_0[\,\vec{t}\,;z] = \sum_{k\geq 1} kt_k\,\frac{\partial}{\partial t_k} - z\,\frac{\partial}{\partial z} + \frac{N-1}{2} \tag{5.88}$$

The eigenvalue equation (5.87) shows that the lowest Virasoro constraint is indeed a true symmetry of the system, in that the operators $\mathbf{Q}$ and $L_0[\,\vec{t}\,;z]$ are simultaneously diagonalizable in the basis of bi-orthonormal Baker-Akhiezer functions. The same is true of all higher Virasoro constraints. This shows precisely what sort of constraints the auxiliary eigenvalue problem (5.36) must possess in order to correctly reconstruct the fermionic τ-function. However, an analogous computation using the translation generator $\frac{\partial}{\partial z}$ on eigenvalue space leads to the result

$$\left(\sum_{k\geq 1} kt_k\,\frac{\partial}{\partial t_{k-1}} - \frac{\partial}{\partial z} - \frac{N+1}{2z}\right)\Psi_n[\,\vec{t}\,;z] = -(n+N)R_n\,\Psi_{n-1}[\,\vec{t}\,;z] \tag{5.89}$$

which shows that there is no corresponding L_{-1} operator which commutes with the Lax operator $\mathbf{Q}$. This means that there is no translational symmetry in the system, and the leading equations of motion at large N are determined instead by the generator L_0 of scale transformations of the system. These equations can be written in a form comparable to those of sections 3 and 4 by writing a consistency condition between the eigenvalue equations (5.36) and (5.87) in the form

$$\Big(L_0[\,\vec{t}\,;z] + n\Big)\Big(\mathbf{Q} - z\Big)\Psi_n[\,\vec{t}\,;z] = 0 \tag{5.90}$$

Expanding (5.90) using (5.87), (5.88) and completeness of the Baker-Akhiezer functions then leads to the system of equations

$$(1+n-m)\,[\mathbf{Q}]_{nm} + \sum_{k\geq 1} kt_k\,\frac{\partial[\mathbf{Q}]_{nm}}{\partial t_k} = 0 \tag{5.91}$$

In particular, setting $n=m$ in (5.91) and using the flow equation (5.31), we have

$$[\mathbf{Q}]_{nn} = \sum_{k\geq 1} kt_k\left(\Big[\mathbf{Q}^k\Big]_{n,n-1} - \Big[\mathbf{Q}^k\Big]_{n+1,n}\right) \tag{5.92}$$

What this means is that it is the operator $\mathbf{P}\,\mathbf{Q}$ (along with its conjugate $\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-1}$) which plays the fundamental role in the dynamics of the adjoint fermion one-matrix model. Thus it is the scale invariance of the system which leads to the fundamental double scaling equations in the large N limit. The translational symmetry generated by the operator $\mathbf{P}$ itself (and its conjugate $\overline{\mathbf{P}}^\dagger\,\mathbf{Q}^{-2}$) is an extra constraint on the theory which plays no immediate role in the continuum limit of the matrix model. By considering the matrix elements $\langle\Phi_n(z)|z\,[z\,\Lambda_m(z)]'\rangle$, it is possible to infer the commutation relation

$\left[\mathbf{Q}\,,\,\overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-1}\right] = \mathbf{Q}$. Using the canonical commutator (5.6), it follows that the fundamental symmetry operator of the system is given by

$$\Pi = \tfrac{1}{2}\,\left(\mathbf{P}\,\mathbf{Q} - \overline{\mathbf{P}}^{\dagger}\,\mathbf{Q}^{-1}\right) \tag{5.93}$$

which together with the Lax operator $\mathbf{Q}$ obeys the non-canonical commutation relation

$$\left[\Pi\,,\,\mathbf{Q}\right] = \mathbf{Q} \tag{5.94}$$

The operator expression (5.94) defines the appropriate string equations of the fermionic matrix model. The upper and lower triangular parts of the scaling operator (5.93) may be computed by using (5.9), (5.51), (5.84) and (5.85) to get

$$\begin{aligned}\Pi_+ &= \frac{1}{2}\sum_{n\geq 0}(n-N)\,\mathbf{E}_{n,n} + \frac{1}{2}\sum_{k\geq 1} k t_k\,\mathbf{Q}_+^k \\ \Pi_- &= -\frac{1}{2}\sum_{k\geq 1} k t_k\,\mathbf{Q}_-^k\end{aligned} \tag{5.95}$$

Substituting (5.95) into the string equation (5.94) and using the KP equation (5.22), we arrive at (5.91).

It is the equation (5.92) which produces the novel Painlevé expansions of the fermionic matrix model. For instance, in the case of the quadratic potential studied in section 4, it is implicit in the manipulations carried out in section 4.2. In fact, the structural form of (5.91) is identical to the so-called "automodel" constraints which arise in the usual Hermitian one-matrix models [28]. On the other hand, in the continuum limit $\mathbf{Q}$ becomes a differential operator of a certain finite degree and the string equations (5.94) can be translated into an equation for pseudo-differential operators which satisfy a generalized KdV hierarchy of flow equations [4]. General solutions of string equations such as (5.94) have been studied before, in the case that the double scaling limit of $\mathbf{Q}$ is described by a Schrödinger operator, with stable, pole-free solutions for the corresponding string susceptibility [35]. We see therefore that the string equations of the fermionic one-matrix model lead to a Painlevé differential equation which has the same structure as that in the usual Hermitian one-matrix models, except that now the automodel form of the equations produce an alternating genus expansion. Moreover, the structural form (5.94) of the string equations allow us to restrict to positive values of the double-scaling variable ξ and thereby obtain pole-free solutions for the free energy of the fermionic matrix model.

In the present case, the details of the continuum limit of the operator $\mathbf{Q}$ are somewhat involved. However, the discrete equations (5.91), (5.92) and (5.94) completely characterize the double scaled equations for the partition function in the large N limit and the ensuing topological expansion. They serve as the starting point for a characterization of the differential hierarchies of the fermionic one-matrix model. From the generic structure of the flow equations obtained in this section, we can expect that the continuum equations satisfied by the partition function of "fermionic quantum gravity" are closely related to the

usual KdV flow structure of two-dimensional quantum gravity in terms of Gelfand-Dikii differential polynomials [4, 5].

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Appendix A Spectral Correlation Functions

In this appendix we will briefly describe some properties of correlators of the fermionic one-matrix model within the formalism of this paper. The symmetries of the matrix integral (1.1) restrict its observables to those which are invariant functions of $\bar\psi\psi$ [9]. Connected correlation functions may be generated by taking derivatives of the free energy with respect to the coupling constants of the potential V,

$$\left\langle \prod_{j=1}^{L} \operatorname{tr}\left(\bar\psi\psi\right)^{p_j} \right\rangle_{\mathrm{F}\,,\,\mathrm{conn}} = \frac{1}{N^L} \prod_{j=1}^{L} p_j \, \frac{\partial}{\partial g_{p_j}} \, \log Z_N \tag{A.1}$$

where the normalized fermionic correlation functions are defined by

$$\left\langle f\left(\bar\psi\psi\right)\right\rangle_{\mathrm{F}} \equiv \frac{1}{Z_N} \int\limits_{\mathrm{Gr}(N)^c} d\psi \; d\bar\psi \; f\left(\bar\psi\psi\right) \; \mathrm{e}^{N \operatorname{tr} V(\bar\psi\psi)} \tag{A.2}$$

and in (A.1) it is understood that, if necessary, a set of auxiliary coupling constants g_k are introduced into the potential V and then set to zero after differentiation. Given the equivalence of the generating function $\log Z_N$ for the connected correlators of the fermionic and unitary matrix models, we therefore also have complete equivalence of their observables,

$$\left\langle \prod_{j=1}^{L} \operatorname{tr}\left(\bar\psi\psi\right)^{p_j} \right\rangle_{\mathrm{F}\,,\,\mathrm{conn}} = \left\langle \prod_{j=1}^{L} \operatorname{tr} U^{p_j} \right\rangle_{\mathrm{U}\,,\,\mathrm{conn}} \tag{A.3}$$

where the normalized unitary correlation functions are defined by

$$\left\langle f(U)\right\rangle_{\mathrm{U}} \equiv \frac{k_N}{Z_N} \int\limits_{U(N)} [dU] \; f(U) \; \mathrm{e}^{N \operatorname{tr}\left(V(U) - \log U\right)} \tag{A.4}$$

The correlation functions in (A.3) may be evaluated as in (2.29)–(2.33). For example, one may readily compute

$$\left\langle \operatorname{tr}\left(\bar\psi\psi\right)^{p}\right\rangle_{\mathrm{F}} = \left\langle \mathrm{e}^{ip\theta} \; \operatorname{tr} U_0^p \right\rangle_{\mathrm{U}} = \frac{k_N}{N Z_N} \left[\int\limits_{SU(N)} [dU_0] \; \operatorname{tr} U_0^p \; \mathrm{e}^{N \operatorname{tr} V(U_0)} \right]_{C_1 = N^2 - p} \tag{A.5}$$

The restriction in (A.5) of the character expansion of $\mathrm{e}^{N\,\mathrm{tr}\,V(U_0)}$ to Young tableaux with N^2-p filled boxes implies that single trace correlators are non-vanishing only for $p \le N^2$. The finiteness of the set of non-vanishing correlators is of course natural as a consequence of the anticommuting property of the matrices ψ and $\bar\psi$, but it is a dramatic result in the unitary matrix model. Thus not only does the Penner interaction term in the unitary matrix model (2.28) capture the finite nature of the perturbation expansion of Z_N, but it also reduces the correlation functions appropriately to reproduce the correct properties of the fermionic correlators at finite N. Note that, geometrically, the operator $\mathrm{tr}(\bar\psi\psi)^p$ inserts, on the dual triangulated surface, a hole with p boundary lengths. The lattice expansion of such operators thereby generates fermionic ribbon graphs which are dual to tesselations of Riemann surfaces with a given number of boundaries. Again there are only finitely many such fat-graphs, because the maximum number of boundaries that a given discretization can have is N^2.

The complete set of observables of the fermionic matrix model can be generated by the joint probability distributions

$$\rho_n(z_1,\dots,z_n) = \frac{(N-n)!}{N!}\,\Big\langle \prod_{k=1}^{n} \mathrm{tr}\,\delta\left(z_k-\bar\psi\psi\right)\Big\rangle_{\mathrm{F}} \tag{A.6}$$

where z_k are points in the complex plane and $1 \le n \le N$. When the correlation function in (A.6) is mapped to the unitary matrix model as described above, the points z_k can be interpreted as eigenvalues of unitary matrices. By diagonalizing the unitary matrices in (A.4), using the identities

$$\Delta(z_1,\dots,z_N) = \det_{i,j}\Big[\Phi_{j-1}(z_i)\Big] \qquad , \qquad \overline{\Delta(z_1,\dots,z_N)} = \det_{i,j}\Big[\Lambda_{j-1}(z_i)\Big] \tag{A.7}$$

which follow from (2.6) and (3.5), and by using (3.7), it is straightforward to derive the determinant representation

$$\rho_n(z_1,\dots,z_n) = \frac{(N-n)!}{N!}\,\Big\langle \prod_{k=1}^{n} \mathrm{tr}\,\delta(z_k-U)\Big\rangle_{\mathrm{U}} = \frac{(N-n)!}{N!}\,\det_{i,j}\Big[\mathcal{K}(z_i,z_j)\Big] \tag{A.8}$$

where $\mathcal{K}(z,z')$ is the spectral kernel which is defined in terms of the orthogonal polynomials as

$$\mathcal{K}(z,z') = \frac{\mathrm{e}^{\frac{N}{2}\left(V(z)+V(z')\right)}}{(z\,z')^{\frac{N+1}{2}}}\,\sum_{n=0}^{N-1}\frac{\Phi_n(z)\,\Lambda_n(z')}{h_n} \tag{A.9}$$

We see therefore that the problem of evaluating correlation functions of the fermionic matrix model reduces to that of determining the spectral kernel (A.9). It is possible to express it in a much simpler form by deriving the appropriate generalization of the Christoffel-Darboux formula [1]. For this, we first need to derive a "mixed" recursion relation between the Φ and Λ polynomials. Consider the bi-orthogonality relations

$$\begin{aligned}\Big\langle \Phi_{n+1}(z) - z\,\Phi_n(z)\,\Big|\,z^{-m}\Big\rangle &= \Big\langle \Phi_{n+1}(z)\,\Big|\,z^{-m}\Big\rangle - \Big\langle \Phi_n(z)\,\Big|\,z^{-(m-1)}\Big\rangle \;=\; 0\\ \Big\langle z^n\,\Lambda_n(z)\,\Big|\,z^{-m}\Big\rangle &= \Big\langle z^{n-m}\,\Big|\,\Lambda_n(z)\Big\rangle \;=\; 0\end{aligned} \tag{A.10}$$

which are valid for $1 \leq m \leq n$. From (A.10) it follows that the two polynomials $\Phi_{n+1}(z) - z\,\Phi_n(z)$ and $z^n\,\Lambda_n(z)$ of degree n are equal up to some constant c_n. Equating the constant terms of these polynomials gives $c_n = p_{n+1,0}$, and we arrive at the mixed three-term recursion relation

$$\Phi_{n+1}(z) = z\,\Phi_n(z) + p_{n+1,0}\,z^n\,\Lambda_n(z) \tag{A.11}$$

In an analogous way, it is possible to derive the recursion relation

$$\Lambda_{n+1}(z) = \frac{1}{z}\,\Lambda_n(z) + l_{n+1,0}\,\frac{1}{z^n}\,\Phi_n(z) \tag{A.12}$$

We now multiply (A.11) through by $\Lambda_n(z')/h_n$ and (A.12) with the index shift $n \to n-1$ and $z \to z'$ through by $z'\,\Phi_n(z)/h_n$, subtract the resulting two equations, and then sum over $n = 1, \ldots, N-1$. This yields

$$\begin{aligned}(z-z')\sum_{n=1}^{N-1}\frac{\Phi_n(z)\,\Lambda_n(z')}{h_n} &= \frac{\Phi_N(z)\,\Lambda_{N-1}(z')}{h_{N-1}} - \frac{\Phi_1(z)}{h_1} \\ &\quad - \sum_{n=1}^{N-1}\frac{1}{h_k}\left[p_{n+1,0}\,z^n\,\Lambda_n(z)\,\Lambda_n(z') - \frac{l_{n,0}}{z'^{n-1}}\,\Phi_n(z)\,\Phi_{n-1}(z')\right]\end{aligned} \tag{A.13}$$

We can iterate the recursion equations (A.11) and (A.12) to get

$$z^k\,\Lambda_k(z) = 1 + z\sum_{j=1}^{k} l_{j,0}\,\Phi_{j-1}(z) \qquad , \qquad \frac{1}{z'^{k-1}}\,\Phi_{k-1}(z') = 1 + \frac{1}{z'}\sum_{j=1}^{k-1} p_{j,0}\,\Lambda_{j-1}(z') \tag{A.14}$$

Substituting (A.14) into (A.13) and comparing the resulting equation with itself under the interchange of arguments $z \leftrightarrow z'$, we arrive after some algebra at

$$\begin{aligned}(z-z')\sum_{n=0}^{N-1}\frac{\Phi_n(z)\,\Lambda_n(z')}{h_n} &= \frac{z'^{N-1}\,\Lambda_{N-1}(z')\,\Phi_N(z) - z^{N-1}\,\Lambda_{N-1}(z)\,\Phi_N(z')}{h_{N-1}\,z'^{N-1}} \\ &\quad + \frac{1-h_0}{h_0}\,(z-z')\end{aligned} \tag{A.15}$$

The expression (A.15) is valid for any pair of complex numbers $z \neq z'$. We can take the $z \to z'$ limit of (A.15) using l'Hospital's rule to get

$$\sum_{n=0}^{N-1}\frac{\Phi_n(z)\,\Lambda_n(z)}{h_n} = \frac{z^{N-1}\,\Lambda_{N-1}(z)\,\Phi'_N(z) - \Phi_N(z)\left(z^{N-1}\,\Lambda_{N-1}(z)\right)'}{h_{N-1}\,z^{N-1}} + \frac{1-h_0}{h_0} \tag{A.16}$$

The identities (A.15) and (A.16) are the fermionic analogs of the Christoffel-Darboux formula for the usual Hermitian one-matrix model orthogonal polynomials.

The Christoffel-Darboux formula allows us to express the spectral kernel (A.9) in terms of orthogonal polynomials with indices n close to the matrix dimension N. Using

the mixed recursion relation (A.11), we obtain

$$\begin{aligned}\mathcal{K}(z,z') &= \frac{\mathrm{e}^{\frac{N}{2}\left(V(z)+V(z')\right)}}{\left(z\,z'\right)^{\frac{N+1}{2}}} \\ &\quad\times\left\{\frac{z'^{-N+1}}{h_{N-1}p_{N,0}\,(z-z')}\left[z\,\Phi_{N-1}(z)\,\Phi_N(z')-z'\,\Phi_{N-1}(z')\,\Phi_N(z)\right]+\frac{1-h_0}{h_0}\right\}\end{aligned} \tag{A.17}$$

It follows that the fermionic matrix model is completely determined by the single set of Φ polynomials, as expected because of the one-matrix nature of the model and the heuristic identification $U \sim \bar\psi\psi$. In particular, for the spectral density we find

$$\begin{aligned}\rho(z) &\equiv \frac{1}{N}\,\mathcal{K}(z,z) \\ &= \frac{\mathrm{e}^{NV(z)}}{Nz^N}\left\{\frac{1}{h_{N-1}p_{N,0}\,z^N}\,\left[z\left(\Phi_N(z)\,\Phi'_{N-1}(z)-\Phi'_N(z)\,\Phi_{N-1}(z)\right)+\Phi_N(z)\,\Phi_{N-1}(z)\right]\right. \\ &\quad\left.+\frac{1-h_0}{h_0}\right\}\end{aligned} \tag{A.18}$$

The formula (A.18) is particularly useful for determining the spectral density, and hence all correlators, of the fermionic matrix model in the large N limit. In that case, factorization and symmetry imply that all connected correlation functions vanish and the large N limit of the model is completely characterized by the set of correlators

$$\left\langle\frac{1}{N}\,\mathrm{tr}\left(\bar\psi\psi\right)^p\right\rangle_{\mathrm{F}} = \int dz\;\rho(z)\,z^p \tag{A.19}$$

where the integral goes over the support of the function $\rho(z)$ in the complex plane. Note that the spectral density is defined *a priori* in the adjoint fermion matrix model by the formula (A.6) [11, 9]. In the present formalism it has a natural interpretation as the probability density of *eigenvalues* in the corresponding unitary one-matrix model. As such, it is generally supported on the unit circle. However, as discussed in the previous subsection, the restriction of the integration contour to the unit circle in the large N limit is not necessary, and the spectral density can be supported generically in the complex plane. These facts explain the general properties of the spectral densities of fermionic matrix models [10, 11, 9], for instance how the pole generated by the Penner interaction requires the support contour of $\rho(z)$ to be adjusted so as to avoid the origin of the complex plane and why the support endpoints are complex-valued.

Appendix B Solution of the Gaussian Model

In this appendix we will illustrate how the formalism of section 3 works in a specific example. We shall consider the example of the Gaussian potential

$$V(z) = z \tag{B.1}$$

for which everything can be obtained explicitly. In this case the recursion relations of section 3.2 are easily solved to give the coefficients

$$\begin{aligned} R_n &= \frac{n}{n+N} \\ P_n &= \frac{n}{N} \\ Q_n &= -\left(1+\frac{n+1}{N}\right) \end{aligned} \tag{B.2}$$

and the normalization constants are

$$h_n = 2\pi i\, N^N\, \frac{n!}{(n+N)!} \tag{B.3}$$

Substituting (B.2) into the three-term recursion relation (3.29), we obtain

$$\Phi_n(z) = \left(z+\frac{n-1+N}{N}\right)\Phi_{n-1}(z) - \frac{(n-1)z}{N}\,\Phi_{n-2}(z) \tag{B.4}$$

To solve the recurrence relation (B.4), we introduce the generating function which is defined as the formal power series

$$\Xi_N(z;s) = \sum_{n=0}^{\infty} \Phi_n(z)\; s^n \tag{B.5}$$

in a variable s, with the boundary condition $\Xi_N(z;0)=1$. The recursion relation (B.4) is then equivalent to a first order inhomogeneous linear differential equation for the function Ξ_N,

$$\left[\frac{s^2}{N}\left(1-sz\right)\frac{\partial}{\partial s} + s\left(z+1-\frac{sz}{N}\right) - 1\right]\Xi_N(z;s) = -1 \tag{B.6}$$

Integrating (B.6) we find that the generating function (B.5) is given by

$$\Xi_N(z;s) = \frac{1}{s}\sum_{k=0}^{\infty}\frac{(N+k-1)!}{(N-1)!}\left(\frac{1}{N}\right)^k\left(\frac{s}{1-sz}\right)^{k+1} \tag{B.7}$$

Expanding the function (B.7) as a power series in s and equating its coefficient of s^n with that of the definition (B.5), we arrive at an explicit form for the polynomials $\Phi_n(z)$,

$$\Phi_n(z) = \sum_{k=0}^{n}\binom{n}{k}\frac{(N+k-1)!}{(N-1)!}\left(\frac{1}{N}\right)^k\; z^{n-k} \tag{B.8}$$

which can be expressed as

$$\Phi_n(z) = N^N\, z^{N+n}\, U[N, n+1+N; Nz] \tag{B.9}$$

where

$$U[a,c;x] = \frac{\pi}{\sin\pi c}\left(\frac{{}_1F_1[a,c;x]}{\Gamma(a-c+1)\Gamma(c)} + \frac{x^{1-c}\;{}_1F_1[a-c+1,2-c;x]}{\Gamma(a)\Gamma(2-c)}\right) \tag{B.10}$$

is a confluent hypergeometric function. The partition function and observables of the Gaussian fermionic one-matrix model are therefore completely determined by a system of confluent hypergeometric polynomials.

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